14. Множення гомотопійних границь. Навколо похідних категорій

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Множення конусів

Let A_1, A_2, A'_1, A'_2 be cochain complexes of $\Bbbk$-modules. Suppose further that, for $i \in \{1, 2\}$, we are given cochain maps $\psi_i : A_i \to A'_i$. There are morphisms $\psi_1 \otimes \mathsf{id} \colon A_1 \otimes A_2 \to A'_1 \otimes A_2$ and $\mathsf{id} \otimes \psi_2 \colon A_1 \otimes A_2 \to A_1 \otimes A'_2$. Combine them to form a single map $\Psi = (\psi_1 \otimes \mathsf{id} \;\; \mathsf{id} \otimes \psi_2) \colon A_1 \otimes A_2 \to (A'_1 \otimes A_2) \oplus (A_1 \otimes A'_2)$. Write the multiplication

$$\begin{aligned}\operatorname{Cone}(\psi_1 : A_1 \to A'_1)[-1] \otimes \operatorname{Cone}(\psi_2 : A_2 \to A'_2)[-1] \\ \to \operatorname{Cone}(\Psi : A_1 \otimes A_2 \to (A'_1 \otimes A_2) \oplus (A_1 \otimes A'_2))[-1]\end{aligned}$$

as the truncation = projection of 4 terms to 3 terms

$$\left(A_1 \oplus A'_1[-1], \begin{pmatrix} d_{A_1} & -\psi_1 \cdot \sigma^{-1} \\ 0 & d_{A'_1[-1]} \end{pmatrix} \right) \otimes \left(A_2 \oplus A'_2[-1], \begin{pmatrix} d_{A_2} & -\psi_2 \cdot \sigma^{-1} \\ 0 & d_{A'_2[-1]} \end{pmatrix} \right) \to$$

$$\left(A_1 \otimes A_2 \to (A'_1 \otimes A_2 \oplus A_1 \otimes A'_2)[-1], \begin{pmatrix} d_{A_1 \otimes A_2} & -(\psi_1 \otimes 1 \;\; 1 \otimes \psi_2) \cdot \sigma^{-1} \\ 0 & d_{(A'_1 \otimes A_2 \oplus A_1 \otimes A'_2)[-1]} \end{pmatrix} \right).$$

Let A_1, A_2, A_3 be cochain complexes of $\mathbb{k}$-modules, and let $\mu : A_1 \otimes A_2 \to A_3$ be a cochain map, which we should think of as the composition. Suppose further that, for $i \in \{1, 2, 3\}$, we are given cochain maps $\phi_i : A_i \to A_i$ such that the square below commutes

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{\mu} & A_3 \\ \downarrow{\scriptstyle \phi_1 \otimes \phi_2} & & \downarrow{\scriptstyle \phi_3} \\ A_1 \otimes A_2 & \xrightarrow{\mu} & A_3. \end{array} \tag{1}$$

Now for $i \in \{1, 2, 3\}$ we define

$$\widetilde{A}_i := \mathrm{Cone}\left(A_i \xrightarrow{\mathsf{id} - \phi_i} A_i \right)[-1].$$

Combine $(\mathsf{id} - \phi_1) \otimes \mathsf{id} \colon A_1 \otimes A_2 \to A_1 \otimes A_2$ and $\mathsf{id} \otimes (\mathsf{id} - \phi_2) \colon A_1 \otimes A_2 \to A_1 \otimes A_2$ to form a single map $\Psi = ((\mathsf{id} - \phi_1) \otimes \mathsf{id} \;\; \mathsf{id} \otimes (\mathsf{id} - \phi_2)) \colon A_1 \otimes A_2 \to (A_1 \otimes A_2) \oplus (A_1 \otimes A_2)$.

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{\Psi} & (A_1 \otimes A_2) \oplus (A_1 \otimes A_2) \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle (\mu, \mu \circ (\phi_1 \otimes \mathsf{id}))} \\ A_3 & \xrightarrow{\mathsf{id} - \phi_3} & A_3. \end{array} \tag{2}$$

commutes because it is equivalent to

$$\begin{array}{ccc} A_1 \otimes A_2 & & \\ \Big\downarrow{\scriptstyle \mu} & \searrow{\scriptstyle [(1-\phi_1)\otimes 1]\cdot\mu + [\phi_1 \otimes (1-\phi_2)]\cdot\mu} & \\ A_3 & \xrightarrow{1-\phi_3} & A_3 \end{array}$$

which commutes since (1) does.

Конус як функтор

Cone is a functor $\mathsf{Cone} : \mathsf{dg}^{\rightarrow} \to \mathsf{dg}$. The left square in

$$\begin{array}{ccccccc} X & \xrightarrow{\Psi} & Y & \xrightarrow{\mathsf{in}_2} & \mathsf{Cone}\,\Psi & \xrightarrow{\mathsf{pr}_1} & X[1] \\ \mu\downarrow & = & \downarrow g & = & \downarrow h & = & \downarrow \mu[1] \\ W & \xrightarrow{f} & Z & \xrightarrow{\mathsf{in}_2} & \mathsf{Cone}\,f & \xrightarrow{\mathsf{pr}_1} & W[1] \end{array}$$

determines a map of cones $h = \begin{pmatrix} \mu[1] & 0 \\ 0 & g \end{pmatrix}$.

There is also a functor $\mathsf{Cone}[-1] : \mathsf{dg}^{\rightarrow} \to \mathsf{dg}$,

$h[-1] = \sigma \cdot h \cdot \sigma^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & g[-1] \end{pmatrix}$. Applied to (2) it gives

$\Theta : \mathrm{Cone}\left(A_1 \otimes A_2 \xrightarrow{\Psi} (A_1 \otimes A_2) \oplus (A_1 \otimes A_2)\right)[-1] \to$

$\mathrm{Cone}\left(A_3 \xrightarrow{\mathsf{id} - \phi_3} A_3\right)[-1]$.

Множення конусів зі зсувом

The composition map $\widetilde{\mu} : \widetilde{\mathrm{A}}_1 \otimes \widetilde{\mathrm{A}}_2 \to \widetilde{\mathrm{A}}_3$ is set to be the composite

$$\begin{array}{c}
\mathrm{Cone}\left(\mathrm{A}_1 \xrightarrow{\mathsf{id} - \phi_1} \mathrm{A}_1\right)[-1] \otimes \mathrm{Cone}\left(\mathrm{A}_2 \xrightarrow{\mathsf{id} - \phi_2} \mathrm{A}_2\right)[-1] \\
\Big\downarrow \text{truncation} \\
\mathrm{Cone}\left(\mathrm{A}_1 \otimes \mathrm{A}_2 \xrightarrow{\Psi} (\mathrm{A}_1 \otimes \mathrm{A}_2) \oplus (\mathrm{A}_1 \otimes \mathrm{A}_2)\right)[-1] \\
\Big\downarrow \Theta \\
\mathrm{Cone}\left(\mathrm{A}_3 \xrightarrow{\mathsf{id} - \phi_3} \mathrm{A}_3\right)[-1].
\end{array}$$

Множення зворотних послідовностей

Suppose now that A_1, A_2 and A_3 are inverse sequences of cochain complexes of $\Bbbk$-modules. That is: for any integer $n > 0$ and for $i \in \{1,2,3\}$ we are given a cochain complex $A_{i,n}$, these come with multiplication maps $\mu_n : A_{1,n} \otimes A_{2,n} \to A_{3,n}$ and with sequence maps $\phi_{i,n} \colon A_{i,n+1} \to A_{i,n}$, and for each n the square below commutes

$$\begin{array}{ccc} A_{1,n+1} \otimes A_{2,n+1} & \xrightarrow{\mu_{n+1}} & A_{3,n+1} \\ \Big\downarrow{\scriptstyle \phi_{1,n}\otimes\phi_{2,n}} & & \Big\downarrow{\scriptstyle \phi_{3,n}} \\ A_{1,n} \otimes A_{2,n} & \xrightarrow{\mu_n} & A_{3,n} \end{array}$$

For $i \in \{1,2,3\}$ define $\widehat{A}_i := \prod_{n>0} A_{i,n}$. The multiplication map $\widehat{\mu} \colon \widehat{A}_1 \otimes \widehat{A}_2 \to \widehat{A}_3$ is the composite

$$\left(\prod_{n>0} A_{1,n}\right) \otimes \left(\prod_{n>0} A_{2,n}\right) \longrightarrow \prod_{n>0} (A_{1,n} \otimes A_{2,n}) \xrightarrow{\prod_{n>0} \mu_n} \prod_{n>0} A_{3,n}.$$

Множення гомотопійних границь

If we let $\phi_i \colon \widehat{A}_i \to \widehat{A}_i$ be the composite

$$\prod_{n>0} A_{i,n} \xrightarrow{\text{projection}} \prod_{n>0} A_{i,n+1} \xrightarrow{\prod_{n>0} \phi_{1,n}} \prod_{n>0} A_{i,n},$$

then the square below commutes

$$\begin{array}{ccc} \widehat{A}_1 \otimes \widehat{A}_2 & \xrightarrow{\widehat{\mu}} & \widehat{A}_3 \\ \Big\downarrow{\scriptstyle \phi_1 \otimes \phi_2} & & \Big\downarrow{\scriptstyle \phi_3} \\ \widehat{A}_1 \otimes \widehat{A}_2 & \xrightarrow{\widehat{\mu}} & \widehat{A}_3 \end{array}$$

Setting

$$\mathsf{holim}\, A_{i,n} = \widetilde{A}_i := \mathrm{Cone}\left(\widehat{A}_i \xrightarrow{\mathsf{id} - \phi_i} \widehat{A}_i\right)[-1],$$

the preceding discussion showed us how to construct the composition $(\widetilde{\mu} : \widetilde{A}_1 \otimes \widetilde{A}_2 \to \widetilde{A}_3) =$
$(\mathsf{holim}\, \mu_n \colon\ \mathsf{holim}\, A_{1,n} \otimes \mathsf{holim}\, A_{2,n} \longrightarrow \mathsf{holim}\, A_{3,n})$.

Let $\mathcal{B}$ be a dg-category. The dg category $\mathcal{B}'$ has the same objects as $\mathcal{B}$, and the dg functor $\mathcal{B} \to \mathcal{B}'$ is the identity on objects. The Hom-complexes in the dg category $\mathcal{B}'$, as well as the dg functor $\mathcal{B} \to \mathcal{B}'$, are specified by giving the cochain map $\mathcal{B}(\mathrm{B}_1, \mathrm{B}_2) \to \mathcal{B}'(\mathrm{B}_1, \mathrm{B}_2)$ for every pair of objects $\mathrm{B}_1, \mathrm{B}_2 \in \mathcal{B}$. We declare this to be the natural cochain map

$$\mathcal{B}(\mathrm{B}_1, \mathrm{B}_2) \longrightarrow \mathsf{holim}\, \mathcal{B}(\mathrm{B}_1, \mathrm{B}_2)$$

where on the right we mean the homotopy limit of the inverse sequence

$$\cdots \longrightarrow \mathcal{B}(\mathrm{B}_1, \mathrm{B}_2) \xrightarrow{\mathsf{id}} \mathcal{B}(\mathrm{B}_1, \mathrm{B}_2) \xrightarrow{\mathsf{id}} \mathcal{B}(\mathrm{B}_1, \mathrm{B}_2)$$

The composition law in the category $\mathcal{B}'$, giving the map

$$\mathcal{B}'(\mathrm{B}_1, \mathrm{B}_2) \otimes \mathcal{B}'(\mathrm{B}_2, \mathrm{B}_3) \longrightarrow \mathcal{B}'(\mathrm{B}_1, \mathrm{B}_3)$$

is as above.

It is obvious that the dg functor $\mathcal{B} \to \mathcal{B}'$ is a quasi-equivalence.

Alberto Canonaco, Amnon Neeman, and Paolo Stellari, Uniqueness of enhancements for derived and geometric categories, 2021, arXiv:2101.04404. §4.3, §4.4 Step 1.