13. Гомотопійна границя.
Навколо похідних категорій

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Виправлення

Proposition

Let D and D′ be homotopy equivalent objects of a dg-category $\mathcal{D}$. Let

$$D \xrightarrow[0]{f} D' \qquad g \circ f = 1_D - d\alpha$$

$$\alpha \underset{-1}{\circlearrowright} D \xleftarrow[0]{g} D' \underset{-1}{\circlearrowleft} \beta \qquad f \circ g = 1_{D'} - d\beta$$

be relevant data.

Then there are α' and δ

$$D \xrightarrow[0]{f} D' \qquad g \circ f = 1_D - d\alpha'$$

$$\alpha' \underset{-1}{\circlearrowright} D \xleftarrow[0]{g} D' \underset{-1}{\circlearrowleft} \beta \qquad f \circ g = 1_{D'} - d\beta$$

$$D \xrightarrow[-2]{\delta} D' \qquad f \circ \alpha' - \beta \circ f = d\delta.$$

Доведення.

We have $\mathrm{f}\circ\alpha-\beta\circ\mathrm{f}\in\mathrm{Z}^{-1}\mathcal{D}(\mathrm{D},\mathrm{D}')$ since

$$\mathrm{d}(\mathrm{f}\circ\alpha-\beta\circ\mathrm{f})=\mathrm{f}\circ(1-\mathrm{g}\circ\mathrm{f})+(\mathrm{f}\circ\mathrm{g}-1)\circ\mathrm{f}=0.$$

$\Rightarrow \mathrm{z}:=\mathrm{g}\circ(\mathrm{f}\circ\alpha-\beta\circ\mathrm{f})\in\mathrm{Z}^{-1}\mathcal{D}(\mathrm{D},\mathrm{D})$,
$\alpha':=\alpha-\mathrm{z}\Rightarrow \mathrm{g}\circ\mathrm{f}=1_{\mathrm{D}}-\mathrm{d}\alpha'$

$$\begin{aligned}&\mathrm{f}\circ\alpha'-\beta\circ\mathrm{f}=\mathrm{f}\circ\alpha-\beta\circ\mathrm{f}-\mathrm{f}\circ\mathrm{g}\circ(\mathrm{f}\circ\alpha-\beta\circ\mathrm{f})\\&=\mathrm{f}\circ\alpha-\beta\circ\mathrm{f}-(1-\mathrm{d}\beta)\circ(\mathrm{f}\circ\alpha-\beta\circ\mathrm{f})=(\mathrm{d}\beta)\circ(\mathrm{f}\circ\alpha-\beta\circ\mathrm{f})\\&=\mathrm{d}[\beta\circ(\mathrm{f}\circ\alpha-\beta\circ\mathrm{f})]=:\mathrm{d}\delta.\end{aligned}$$

□

Зворотні системи

Let C be a category. If the ordered set is N={1,2,3,...} with the usual ordering, an inverse system (with values in the category C) over N is often simply called an inverse system.
It consists quite simply of a pair $(\mathrm{M_i}, \mathrm{f_{ii'}})$ where each $\mathrm{M_i}$, i∈N is an object of C, and for each i>i′, i,i′∈N a morphism
$\mathrm{f_{ii'}} : \mathrm{M_i} \to \mathrm{M_i'}$ such that moreover $\mathrm{f_{i'i''}} \circ \mathrm{f_{ii'}} = \mathrm{f_{ii''}}$ whenever this makes sense.
It is clear that in fact it suffices to give the morphisms
$\mathrm{M}_2 \to \mathrm{M}_1$, $\mathrm{M}_3 \to \mathrm{M}_2$, and so on. Hence an inverse system is frequently pictured as follows $\mathrm{M}_1 \xleftarrow{\phi_2} \mathrm{M}_2 \xleftarrow{\phi_3} \mathrm{M}_3 \xleftarrow{\phi_4} \dots$
Moreover, we often omit the transition maps ϕ_i from the notation and we simply say “let $(\mathrm{M_i})$ be an inverse system”.
The collection of all inverse systems with values in C forms a category with the obvious notion of morphism.

If C is an additive category, then the category of inverse systems with values in C is an additive category.
If C is an abelian category, then the category of inverse systems with values in C is an abelian category.
A sequence $(K_i) \to (L_i) \to (M_i)$ of inverse systems is exact if and only if each $K_i \to L_i \to M_i$ is exact.
The limit of such an inverse system is denoted $\lim M_i$, or $\lim_i M_i$.
If C is the category of abelian groups (or sets), then the limit always exists and in fact can be described as follows

$$\lim_i M_i = \{(x_i) \in \prod M_i \mid \phi_i(x_i) = x_{i-1}, i = 2, 3, \ldots\}.$$

However, given a short exact sequence

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

of inverse systems of abelian groups it is not always the case that the associated system of limits is exact. In order to discuss this further we introduce the following notion.

Definition (Умова Мітага–Лефлера)

Let C be an abelian category. We say the inverse system (A_i) satisfies the Mittag–Leffler condition, or for short is ML, if for every i there exists a c=c(i)≥i such that for all k≥c

$$\mathrm{Im}(A_k \to A_i) = \mathrm{Im}(A_c \to A_i).$$

It turns out that the Mittag-Leffler condition is good enough to ensure that the lim-functor is exact, provided one works within the abelian category of abelian groups, or abelian sheaves, etc.

Example

If (A_i, ϕ_{ji}) is a directed inverse system of sets or of modules and the maps ϕ_{ji} are surjective, then clearly the system is Mittag–Leffler. Conversely, suppose (A_i, ϕ_{ji}) is Mittag–Leffler. Let $A'_i \subset A_i$ be the stable image of $\phi_{ji}(A_j)$ for j≥i. Then $\phi_{ji}|A'_j : A'_j \to A'_i$ is surjective for j≥i and $\lim A_i = \lim A'_i$. Hence the limit of the Mittag–Leffler system (A_i, ϕ_{ji}) can also be written as the limit of a directed inverse system over I with surjective maps.

Непорожність границі системи Мітага–Лефлера

Lemma

Let (A_i, ϕ_{ji}) be a directed inverse system over I. Suppose I is countable. If (A_i, ϕ_{ji}) is Mittag–Leffler and the A_i are nonempty, then $\lim A_i$ is nonempty.

Доведення.

Let $i_1, i_2, i_3, \ldots$ be an enumeration of the elements of I. Define inductively a sequence of elements $j_n \in I$ for n=1,2,3,... by the conditions: $j_1 = i_1$, and $j_n \geq i_n$ and $j_n > j_m$ for m<n. Then the sequence j_n is increasing and forms a cofinal subset of I. Hence we may assume $I = \{1, 2, 3, \ldots\}$.
So by previous Example we are reduced to showing that the limit of an inverse system of non-empty sets with surjective maps indexed by the positive integers is non-empty. This is obvious. □

Система Мітага–Лефлера і коротка точна послідовність границь

Lemma

Let $0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$ be an exact sequence of directed inverse systems of abelian groups over I. Suppose I is countable. If (A_i) is Mittag–Leffler, then $0 \to \lim A_i \to \lim B_i \to \lim C_i \to 0$ is exact.

Доведення. Taking limits of directed inverse systems is left exact, hence we only need to prove surjectivity of $\lim B_i \to \lim C_i$. So let $(c_i) \in \lim C_i$. For each $i \in I$, let $E_i = g_i^{-1}(c_i)$, which is nonempty since $g_i : B_i \to C_i$ is surjective. The system of maps $\phi_{ji} : B_j \to B_i$ for (B_i) restrict to maps $E_j \to E_i$ which make (E_i) into an inverse system of nonempty sets.

It is enough to show that (E_i) is Mittag-Leffler. For then previous Lemma would show $\lim E_i$ is nonempty, and taking any element of $\lim E_i$ would give an element of $\lim B_i$ mapping to (c_i).

By the injection $f_i : A_i \to B_i$ we will regard A_i as a subset of B_i. Since (A_i) is Mittag-Leffler, if $i \in I$ then there exists $j \geq i$ such that $\phi_{ki}(A_k) = \phi_{ji}(A_j)$ for $k \geq j$. We claim that also $\phi_{ki}(E_k) = \phi_{ji}(E_j)$ for $k \geq j$. Always $\phi_{ki}(E_k) \subset \phi_{ji}(E_j)$ for $k \geq j$.
For the reverse inclusion let $e_j \in E_j$, and we need to find $x_k \in E_k$ such that $\phi_{ki}(x_k) = \phi_{ji}(e_j)$.
Let $e'_k \in E_k$ be any element, and set $e'_j = \phi_{kj}(e'_k)$. Then $g_j(e_j - e'_j) = c_j - c_j = 0$, hence $e_j - e'_j = a_j \in A_j$.
Since $\phi_{ki}(A_k) = \phi_{ji}(A_j)$, there exists $a_k \in A_k$ such that $\phi_{ki}(a_k) = \phi_{ji}(a_j)$. Hence

$$\phi_{ki}(e'_k + a_k) = \phi_{ji}(e'_j) + \phi_{ji}(a_j) = \phi_{ji}(e_j),$$

so we can take $x_k = e'_k + a_k$. $\square$

Lemma

Let $0 \to (A_i) \to (B_i) \to (C_i) \to 0$ be a short exact sequence of inverse systems of abelian groups. Then
In any case the sequence $0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i$ is exact.
If (B_i) is ML, then also (C_i) is ML.
If (A_i) is ML, then $0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i \to 0$ is exact.

Proof. (1) $\lim : \mathrm{Ab}^{I^{op}} \to \mathrm{Ab}$ is right adjoint to $\mathrm{const} : \mathrm{Ab} \to \mathrm{Ab}^{I^{op}}, X \mapsto (X)_i$.
(2) follows from surjectivity of all $g_i : B_i \to C_i$: $\forall i\ \exists j \geq i\ \forall k \geq j$

$$\begin{array}{ccc} B_k & \xrightarrow{g_k} & C_k \\ \downarrow & & \downarrow \\ B_j & \xrightarrow{g_j} & C_j \\ \downarrow & & \downarrow \\ B_i & \xrightarrow{g_i} & C_i \end{array}$$

$\mathrm{Im}(C_k \to C_i) = g_i(\mathrm{Im}(B_k \to B_i)) = g_i(\mathrm{Im}(B_j \to B_i)) = \mathrm{Im}(C_j \to C_i)$.

□

Lemma

Let

$$(A_i) \to (B_i) \to (C_i) \to (D_i)$$

be an exact sequence of inverse systems of abelian groups. If the system (A_i) is ML, then the sequence

$$\lim_i B_i \to \lim_i C_i \to \lim_i D_i$$

is exact.

Доведення.

Let $Z_i = \mathrm{Ker}(C_i \to D_i)$ and $I_i = \mathrm{Im}(A_i \to B_i)$. Then $\lim Z_i = \mathrm{Ker}(\lim C_i \to \lim D_i)$ and we get a short exact sequence of systems

$$0 \to (I_i) \to (B_i) \to (Z_i) \to 0$$

Moreover, by previous Lemma we see that (I_i) has (ML), thus another application of previous Lemma shows that $\lim B_i \to \lim Z_i$ is surjective which proves the lemma. □

Правий похідний функтор

In this section $\mathscr{C}$ and $\mathscr{C}'$ will denote two abelian categories, and $F : \mathscr{C} \to \mathscr{C}'$ an additive functor.

We shall denote by Q the natural functor $\mathbf{K}^+(\mathscr{C}) \to \mathbf{D}^+(\mathscr{C})$ or $\mathbf{K}^+(\mathscr{C}') \to \mathbf{D}^+(\mathscr{C}')$.

Definition 1.8.1. *Let $T : \mathbf{D}^+(\mathscr{C}) \to \mathbf{D}^+(\mathscr{C}')$ be a functor of triangulated categories, and let s be a morphism of functors:*

$$s : Q \circ \mathbf{K}^+(F) \to T \circ Q \ ,$$

where $\mathbf{K}^+(F) : \mathbf{K}^+(\mathscr{C}) \to \mathbf{K}^+(\mathscr{C}')$ is the functor naturally associated to F. Assume that for any functor of triangulated categories $G : \mathbf{D}^+(\mathscr{C}) \to \mathbf{D}^+(\mathscr{C}')$, the morphism:

$$\mathrm{Hom}(T, G) \xrightarrow[s]{} \mathrm{Hom}(Q \circ \mathbf{K}^+(F), G \circ Q)$$

is an isomorphism.

Then (T, s), which is unique up to isomorphism, is called the right derived functor of F, and denoted RF. The functor $H^n \circ RF$, also denoted R^nF, is called the n-th derived functor of F.

Let us give a useful criterium which ensures the existence of RF. From now on and until Proposition 1.8.7, we assume F is left exact.

F-ін'єктивна підкатегорія

Definition 1.8.2. *A full additive subcategory $\mathscr{I}$ of $\mathscr{C}$ is called injective with respect to F (or F-injective, for short), if :*

(1.7.5) *for any $X \in \mathrm{Ob}(\mathscr{C})$, there exists $X' \in \mathrm{Ob}(\mathscr{I})$ and an exact sequence $0 \to X \to X'$.*

(ii) *if $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathscr{C}$, and if X' and X are in $\mathrm{Ob}(\mathscr{I})$, then X'' is also in $\mathrm{Ob}(\mathscr{I})$,*

(iii) *if $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathscr{C}$, and if X', X, X'', are in $\mathrm{Ob}(\mathscr{I})$, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.*

Note that under conditions (i) and (ii), the condition (iii) is equivalent to the similar condition in which one only assumes $X' \in \mathrm{Ob}(\mathscr{I})$, because of the assumption that F is left exact.

Let $\mathscr{I}$ be F-injective. Then one can check easily that F transforms objects of $\mathbf{K}^+(\mathscr{I})$ quasi-isomorphic to zero into objects of $\mathbf{K}^+(\mathscr{C}')$ satisfying the same property

$$
\begin{array}{ccccccccccccc}
0 \longrightarrow & \mathrm{I}_0 & & \longrightarrow & & \mathrm{I}_1 & \longrightarrow & \mathrm{I}_2 & \longrightarrow & \mathrm{I}_3 & \longrightarrow & \mathrm{I}_4 & \longrightarrow \mathrm{I}_5 \\
& & \| & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \\
& & \mathrm{I}_0 & & & \mathrm{K}_1 & & \mathrm{K}_2 & & \mathrm{K}_3 & & \mathrm{K}_4 \in \mathcal{I} &
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
0 \longrightarrow & \mathrm{FI}_0 & & \longrightarrow & & \mathrm{FI}_1 & \longrightarrow & \mathrm{FI}_2 & \longrightarrow & \mathrm{FI}_3 & \longrightarrow & \mathrm{FI}_4 & \longrightarrow \mathrm{FI}_5 \\
& & \| & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \\
& & \mathrm{FI}_0 & & & \mathrm{FK}_1 & & \mathrm{FK}_2 & & \mathrm{FK}_3 & & \mathrm{FK}_4 &
\end{array}
$$

Існування правого похідного функтора

property. Therefore the composition of functors

$$\mathbf{K}^+(\mathscr{I}) \xrightarrow{\mathbf{K}^+(F)} \mathbf{K}^+(\mathscr{C}') \longrightarrow \mathbf{D}^+(\mathscr{C}')$$

factors through $\mathbf{K}^+(\mathscr{I})/\mathscr{N} \cap \mathrm{Ob}(\mathbf{K}^+(\mathscr{I}))$ where $\mathscr{N}$ is given by acyclic complexes. Since $\mathbf{K}^+(\mathscr{I})/\mathscr{N} \cap \mathrm{Ob}(\mathbf{K}^+(\mathscr{I}))$ is equivalent to $\mathbf{D}^+(\mathscr{C})$ by Proposition 1.7.7, we obtain:

Proposition 1.8.3. *Assume there exists an F-injective subcategory $\mathscr{I}$ of $\mathscr{C}$. Then the functor from $\mathbf{K}^+(\mathscr{I})/\mathscr{N} \cap \mathrm{Ob}(\mathbf{K}^+(\mathscr{I}))$ to $\mathbf{D}^+(\mathscr{C}')$ constructed above is the right derived functor of F.*

Remark 1.8.4. It follows from the universal property of RF that the preceding construction does not depend on $\mathscr{I}$.

Remark 1.8.5. Let $\mathscr{I}$ be the full subcategory of injective objects of $\mathscr{C}$ and assume $\mathscr{C}$ has enough injectives, (i.e: (1.7.5) is satisfied). Then $\mathscr{I}$ is F-injective with respect to any left exact functor F, since any sequence in $\mathscr{I}$ splits, (cf. Exercise I.5). In particular RF always exists in this case.

Гомотопійна границя

In a triangulated category there is a notion of derived limit.

Definition

Let $\mathcal{D}$ be a triangulated category. Let $(K_n, f_n : K_n \to K_{n-1})$ be an inverse system of objects of $\mathcal{D}$. We say an object K is a derived limit, or a homotopy limit of the system (K_n) if the product $\prod K_n$ exists and there is a distinguished triangle

$$K \to \prod K_n \to \prod K_n \to K[1]$$

where the map $\prod K_n \to \prod K_n$ is given by $(k_n) \mapsto (k_n - f_{n+1}(k_{n+1}))$. If this is the case, then we sometimes indicate this by the notation $K = R\lim K_n$.

By TR3 a derived limit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived limit $R\lim K_n$ exists as soon as $\prod K_n$ exists. The derived category $D(Ab)$ of the category of abelian groups is an example of a triangulated category where all derived limits exist.

Lemma

Let $\mathcal{A}$ be an abelian category with countable products and enough injectives. Let $(\mathrm{K_n})$ be an inverse system of $\mathrm{D}^+(\mathcal{A})$. Then $\mathrm{R}\lim \mathrm{K_n}$ exists.

Доведення.

It suffices to show that $\prod \mathrm{K_n}$ exists in $\mathrm{D}(\mathcal{A})$. For every n we can represent $\mathrm{K_n}$ by a bounded below complex $\mathrm{I}_\mathrm{n}^\bullet$ of injectives. Then $\prod \mathrm{K_n}$ is represented by $\prod \mathrm{I}_\mathrm{n}^\bullet$. □

Lemma

The functor $\mathsf{lim} : \mathrm{Ab}^{\mathbb{N}^{\mathsf{op}}} \to \mathrm{Ab}$ has a right derived functor

$$\mathrm{R}\,\mathsf{lim} : \mathrm{D}(\mathrm{Ab}^{\mathbb{N}^{\mathsf{op}}}) \longrightarrow \mathrm{D}(\mathrm{Ab})$$

As usual we set $\mathrm{R}^p \,\mathsf{lim}(K) = \mathrm{H}^p(\mathrm{R}\,\mathsf{lim}(K))$. Moreover, we have

1. for any (A_n) in $\mathrm{Ab}^{\mathbb{N}^{\mathsf{op}}}$ we have $\mathrm{R}^p \,\mathsf{lim}\, A_n = 0$ for $p>1$,
2. the object $\mathrm{R}\,\mathsf{lim}\, A_n$ of $\mathrm{D}(\mathrm{Ab})$ is represented by the complex

   $$\prod A_n \to \prod A_n, (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$

   sitting in degrees 0 and 1,
3. if (A_n) is Mittag–Leffler, then $\mathrm{R}^1 \,\mathsf{lim}\, A_n = 0$, i.e., (A_n) is right acyclic for lim,
4. every $K^\bullet \in \mathrm{D}(\mathrm{Ab}^{\mathbb{N}^{\mathsf{op}}})$ is quasi-isomorphic to a complex whose terms are right acyclic for lim, and
5. if each $K^p = (K_n^p)$ is right acyclic for lim, i.e., of $\mathrm{R}^1 \,\mathsf{lim}_n\, K_n^p = 0$, then $\mathrm{R}\,\mathsf{lim}\, K$ is represented by the complex whose term in degree p is $\mathsf{lim}_n\, K_n^p$.

Proof. Let (A_n) be an arbitrary inverse system. Let (B_n) be the inverse system with

$$B_n = A_n \oplus A_{n-1} \oplus \ldots \oplus A_1$$

and transition maps given by projections. Let $A_n \to B_n$ be given by $(1, f_n, f_{n-1} \circ f_n, \ldots, f_2 \circ \ldots \circ f_n)$ where $f_i : A_i \to A_{i-1}$ are the transition maps. In this way we see that every inverse system is a subobject of a ML system. It follows that every ML system is right acyclic for lim, i.e., (3) holds. This already implies that RF is defined on $D^+(\mathrm{Ab}^{\mathbb{N}^{\mathrm{op}}})$. Set $C_n = A_{n-1} \oplus \ldots \oplus A_1$ for $n>1$ and $C_1 = 0$ with transition maps given by projections as well.
Then there is a short exact sequence of inverse systems $0 \to (A_n) \to (B_n) \to (C_n) \to 0$ where $B_n \to C_n$ is given by $(x_i) \mapsto (x_i - f_{i+1}(x_{i+1}))$. Since (C_n) is ML as well, we conclude that (2) holds which also implies (1).
Finally, this implies that $R\,\mathrm{lim}$ is in fact defined on all of $D(\mathrm{Ab}^{\mathbb{N}^{\mathrm{op}}})$. In fact, one proceeds by proving assertions (4) and (5). □

Let $\mathcal{S}$ be a triangulated category. Suppose X_i, $i \in \mathbb{N}$, is a sequence of objects in $\mathcal{S}$, together with maps $f_i : X_{i+1} \to X_i$. Then $\forall n \in \mathbb{N}$ there is a split exact sequence in $\mathcal{S}$

$$0 \to X_{n+1} \xrightarrow{q} \prod_{i=1}^{n+1} X_i \xrightarrow{\mathsf{pr} - \mathrm{shift}} \prod_{i=1}^{n} X_i \to 0,$$

$$\mathrm{shift} = \left(\prod_{i=1}^{n+1} X_i \xrightarrow{\mathsf{pr}} \prod_{i=2}^{n+1} X_i \xrightarrow{\prod_{i=1}^{n} f_i} \prod_{i=1}^{n} X_i \right),$$

$$\mathsf{pr} - \mathrm{shift} = \begin{pmatrix} 1 & & & & \\ -f_1 & 1 & & & \\ & -f_2 & 1 & & 0 \\ 0 & & \ddots & \ddots & \\ & & & -f_{n-1} & 1 \\ & & & & -f_n \end{pmatrix},$$

$$q = (f_n \dots f_1, \dots, f_n f_{n-1}, f_n, 1).$$

Splitting is determined by $\mathsf{pr}_{n+1} : \prod_{i=1}^{n+1} X_i \to X_{n+1}$.

The diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_{n+1} & \xrightarrow{q} & \prod_{i=1}^{n+1} X_i & \xrightarrow{\mathsf{pr}-\text{shift}} & \prod_{i=1}^{n} X_i & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f_n} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & X_n & \xrightarrow{q} & \prod_{i=1}^{n} X_i & \xrightarrow{\mathsf{pr}-\text{shift}} & \prod_{i=1}^{n-1} X_i & \longrightarrow & 0
\end{array}$$

If $\mathcal{S} = D(\mathcal{A})$, where abelian category satisfies AB5*), the filtered limit of rows would be an exact sequence in $C(\mathcal{A})$

$$0 \to \lim_{i \in \mathbb{N}} X_i \longrightarrow \prod_{i=1}^{\infty} X_i \xrightarrow{1-\text{shift}} \prod_{i=1}^{\infty} X_i \to 0,$$

However, Ab and R-mod do not satisfy AB5*).

For any chain map $\mathrm{f} : \mathrm{X} \to \mathrm{Y}$ there are

$$\mathsf{Cone}(-\mathrm{f} : \mathrm{X} \to \mathrm{Y}) = \left(\mathrm{X}[1] \oplus \mathrm{Y}, \begin{pmatrix} \mathrm{d}_{\mathrm{X}[1]} & -\sigma^{-1} \cdot \mathrm{f} \\ 0 & \mathrm{d}_{\mathrm{Y}} \end{pmatrix}\right),$$

$$\begin{aligned} &\mathsf{Cone}(\mathrm{Y} \to \mathsf{Cone}(-\mathrm{f} : \mathrm{X} \to \mathrm{Y})) \\ &\qquad = \left(\mathrm{Y}[1] \oplus \mathrm{X}[1] \oplus \mathrm{Y}, \begin{pmatrix} \mathrm{d}_{\mathrm{Y}[1]} & 0 & \sigma^{-1} \\ 0 & \mathrm{d}_{\mathrm{X}[1]} & -\sigma^{-1} \cdot \mathrm{f} \\ 0 & 0 & \mathrm{d}_{\mathrm{Y}} \end{pmatrix}\right), \end{aligned}$$

$$\begin{aligned} &\mathrm{Z} = \mathsf{Cone}(\mathrm{Y} \to \mathsf{Cone}(-\mathrm{f} : \mathrm{X} \to \mathrm{Y}))[-1] \\ &\qquad = \left(\mathrm{Y} \oplus \mathrm{X} \oplus \mathrm{Y}[-1], \begin{pmatrix} \mathrm{d}_{\mathrm{Y}} & 0 & -\sigma^{-1} \\ 0 & \mathrm{d}_{\mathrm{X}} & \mathrm{f} \cdot \sigma^{-1} \\ 0 & 0 & \mathrm{d}_{\mathrm{Y}[-1]} \end{pmatrix}\right), \end{aligned}$$

$$\mathrm{d}_{\mathrm{Y}[-1]} = -\sigma \cdot \mathrm{d}_{\mathrm{Y}} \cdot \sigma^{-1}.$$

$$\mathrm{X} \underset{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\overset{(\mathrm{f}\ 1\ 0)}{\rightleftarrows}} \left(\mathrm{Y} \oplus \mathrm{X} \oplus \mathrm{Y}[-1], \begin{pmatrix} \mathrm{d_Y} & 0 & -\sigma^{-1} \\ 0 & \mathrm{d_X} & \mathrm{f} \cdot \sigma^{-1} \\ 0 & 0 & \mathrm{d_{Y[-1]}} \end{pmatrix}\right) \xrightarrow{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \mathrm{Y}.$$

Morphisms on the left are homotopy inverse to each other since

$$\begin{pmatrix} \mathrm{f} & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1,$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{f} & 1 & 0 \end{pmatrix} = 1_\mathrm{Z} + \mathrm{hd_Z} + \mathrm{d_Z h},$$

$$\text{where } \mathrm{h} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix} : \mathrm{Z} \to \mathrm{Z}, \qquad \deg \mathrm{h} = -1.$$

The map f decomposes into homotopy equivalence and a fibration (surjection in all degrees)

$$f = \left(X \xrightarrow[\sim]{(f\ 1\ 0)} Z \xrightarrow{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} Y \right).$$

Iterating this procedure we can replace the sequence (f_i) of chain maps of complexes of abelian groups

$$\begin{array}{ccccccccc} \xrightarrow{f_4} & X_4 & \xrightarrow{f_3} & X_3 & \xrightarrow{f_2} & X_2 & \xrightarrow{f_1} & X_1 \\ = & h_4 \downarrow \wr & = & h_3 \downarrow \wr & = & h_2 \downarrow \wr & = & h_1 \| \\ \xrightarrow{g_4} & Z_4 & \xrightarrow{g_3} & Z_3 & \xrightarrow{g_2} & Z_2 & \xrightarrow{g_1} & Z_1 \end{array} \quad (1)$$

with a sequence (g_i) of fibrations such that the vertical maps (h_i) are homotopy equivalences.
By definition, in the sense of model categories

$$\mathsf{holim}_i(f_i) = \mathsf{holim}_i(g_i) = \lim_i(g_i).$$

Since the sequence (g_i) is Mittag–Leffler we have a short exact sequence of complexes

$$0 \to \lim_{i \in \mathbb{N}} (g_i) \longrightarrow \prod_{i=1}^{\infty} Z_i \xrightarrow{1-\text{shift}} \prod_{i=1}^{\infty} Z_i \to 0,$$

which implies that in the sense of triangulated categories $K' = \mathsf{holim}_i(g_i)$ comes from a triangle in $D(Ab)$

$$K' = \lim_{i \in \mathbb{N}} (g_i) \to \prod_i Z_i \xrightarrow{1-\text{shift}} \prod_i Z_i \to K'[1]$$

isomorphic in $D(Ab)$ to

$$K = \mathsf{holim}_{i \in \mathbb{N}}(f_i) \to \prod_i X_i \xrightarrow{1-\text{shift}} \prod_i X_i \to K[1].$$

Hence, in the sense of triangulated categories $K = \mathsf{holim}_{i \in \mathbb{N}}(f_i) \cong K' = \mathsf{lim}_{i \in \mathbb{N}}(g_i)$ in $D(Ab)$. The same conclusion for any diagram (1) with quasi-isomorphisms h_i and fibrations g_i. Thus, the two approaches to holim agree.

- The Stacks project 12.31 Inverse systems
- The Stacks project 10.86 Mittag-Leffler systems
- The Stacks project 13.34 Derived limits
- The Stacks project Lemma 15.85.1
- Alberto Canonaco, Amnon Neeman, and Paolo Stellari, Uniqueness of enhancements for derived and geometric categories, 2021, arXiv:2101.04404. §3.3
- Masaki Kashiwara and Pierre Schapira, Sheaves on manifolds, Grundlehren der mathematischen Wissenschaften, vol. 292, Springer-Verlag, Berlin, New York, 1990. Def 1.8.1, Def 1.8.2, Prop 1.8.3