9. Ідемпотенти в триангульованій категорії. Навколо похідних категорій

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Абелева категорія

Абелева категорія є адитивною категорією, яка задовольняє аксіомам:
AB1) У будь-якого морфізму $f : A \to B$ існує ядро $\mathsf{ker}\, f : \mathsf{Ker}\, f \to A$ й коядро $\mathsf{coker}\, f : B \to \mathsf{Coker}\, f$.
AB2) Для будь-якого морфізму $f : A \to B$ канонічний морфізм $\mathsf{coker}\,\mathsf{ker}\, f = \mathrm{coim}(f) \to \mathrm{im}(f) = \mathsf{ker}\,\mathsf{coker}\, f$ є ізоморфізмом.

Corollaire 1.2.5. *La somme directe de deux triangles distingués est un triangle distingué.*

Soient :

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \quad ,$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$$

deux triangles distingués et soit de plus :

$$X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \xrightarrow{m} L \xrightarrow{n} (X \oplus X')[1]$$

un triangle distingué contenant le morphisme $u \oplus u'$ (TRI). Les diagrammes :

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & Y \\ {\scriptstyle \begin{pmatrix} \mathsf{id}_X \\ 0 \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} \mathsf{id}_Y \\ 0 \end{pmatrix}} \\ X \oplus X' & \xrightarrow{\quad u \oplus u' \quad} & Y \oplus Y' \end{array} \quad ,$$

$$\begin{array}{ccc} X' & \xrightarrow{\quad u' \quad} & Y' \\ {\scriptstyle \begin{pmatrix} 0 \\ \mathsf{id}_{X'} \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} 0 \\ \mathsf{id}_{Y'} \end{pmatrix}} \\ X \oplus X' & \xrightarrow{\quad u \oplus u' \quad} & Y \oplus Y' \end{array}$$

sont commutatifs et, par suite, s'insèrent en vertu de (TRIII) dans des diagrammes commutatifs :

$$\begin{array}{ccccccc}
X & \xrightarrow{\quad u \quad} & Y & \xrightarrow{\quad v \quad} & Z & \xrightarrow{\quad w \quad} & X[1] \\
\Big\downarrow {\scriptstyle \begin{pmatrix} \mathsf{id}_X \\ 0 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} \mathsf{id}_Y \\ 0 \end{pmatrix}} & & \Big\downarrow {\scriptstyle f} & & \Big\downarrow {\scriptstyle \begin{pmatrix} \mathsf{id}_{X[1]} \\ 0 \end{pmatrix}} \\
X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \xrightarrow{\quad m \quad} & L & \xrightarrow{\quad n \quad} & (X \oplus X')[1]
\end{array} \quad ,$$

$$\begin{array}{ccccccc}
X' & \xrightarrow{\quad u' \quad} & Y' & \xrightarrow{\quad v' \quad} & Z' & \xrightarrow{\quad w' \quad} & X'[1] \\
\Big\downarrow {\scriptstyle \begin{pmatrix} 0 \\ \mathsf{id}_{X'} \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 0 \\ \mathsf{id}_{Y'} \end{pmatrix}} & & \Big\downarrow {\scriptstyle f'} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 0 \\ \mathsf{id}_{X'[1]} \end{pmatrix}} \\
X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \xrightarrow{\quad m \quad} & L & \xrightarrow{\quad n \quad} & (X \oplus X')[1]
\end{array} \quad .$$

On en déduit que le diagramme ci-après est commutatif :

$$\begin{array}{ccccccc}
X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \xrightarrow{v \oplus v'} & Z \oplus Z' & \xrightarrow{w \oplus w'} & (X \oplus X')[1] \\
\Big\downarrow {\scriptstyle \mathsf{id}} & & \Big\downarrow {\scriptstyle \mathsf{id}} & & \Big\downarrow {\scriptstyle (f, f')} & & \Big\downarrow {\scriptstyle \mathsf{id}} \\
X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \xrightarrow{\quad m \quad} & L & \xrightarrow{\quad n \quad} & (X \oplus X')[1]
\end{array} \quad .$$

Pour démontrer le corollaire, il suffit de montrer que le morphisme :

$$(f, f') : Z \oplus Z' \longrightarrow L$$

est un isomorphisme. Il suffit donc de montrer que, pour tout objet M, le morphisme :

$$\mathsf{Hom}_{\mathcal{D}}(M, (f, f'))$$

est un isomorphisme de groupes abeliens; ceci résulte immédiatement de (1.2.1) et du lemme des cinq.

Same proof works for coproduct (direct sum) over an infinite set, if it exists in a given triangulated category $\mathcal{T}$. Subtlety: natural bijections

$$\begin{aligned}\mathcal{T}((\coprod_{i\in I} X_i)[1], Y[1]) &\cong \mathcal{T}(\coprod_{i\in I} X_i, Y) \cong \prod_{i\in I} \mathcal{T}(X_i, Y)\\ &\cong \prod_{i\in I} \mathcal{T}(X_i[1], Y[1]) \cong \mathcal{T}(\coprod_{i\in I}(X_i[1]), Y[1])\end{aligned}$$

imply natural bijection $\alpha : \coprod_{i\in I}(X_i[1]) \cong (\coprod_{i\in I} X_i)[1]$.

Кодобуток виділених трикутників

In particular,

$$\mathsf{in}_j[1] = \left(X_j[1] \xrightarrow{\mathsf{in}_j} \coprod_{i\in I}(X_i[1]) \xrightarrow[\cong]{\alpha} (\coprod_{i\in I} X_i)[1]\right).$$

Coproduct of exact triangles $X_j \xrightarrow{u_j} Y_j \xrightarrow{v_j} Z_j \xrightarrow{w_j} X_j[1]$ is defined as the upper row of the commutative diagram

$$\begin{array}{ccccccccc}
\coprod_{j\in I} X_j & \xrightarrow{\coprod u_j} & \coprod_{j\in I} Y_j & \xrightarrow{\coprod v_j} & \coprod_{j\in I} Z_j & \xrightarrow{\coprod w_j} & \coprod_{j\in I}(X_j[1]) & \xrightarrow[\cong]{\alpha} & (\coprod_{j\in I} X_j)[1] \\
(\mathsf{in}_j) \downarrow = 1 & & (\mathsf{in}_j) \downarrow = 1 & & \downarrow (f_j) & & & & \downarrow 1 \\
\coprod_{i\in I} X_i & \xrightarrow{\coprod u_i} & \coprod_{i\in I} Y_i & \xrightarrow{m} & L & & \xrightarrow{n} & & (\coprod_{i\in I} X_i)[1]
\end{array}$$

So it is exact.

1.6. LEMMA. *Let L be a pre-triangulated category, i.e., a category satisfying all but the octahedron axiom* (TR4). *A triangle*

$$A \oplus A' \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}} T(A) \oplus T(A')$$

is exact if and only if $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ *and* $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} T(A')$ *are both exact.*

LEMMA 1.1. ***Let $\mathscr{A}$ be an abelian category satisfying AB3 (there exist arbitrary direct sums). Then the category K($\mathscr{A}$) of chain complexes over and chain homotopy equivalence classes of maps also has direct sums, and direct sums of triangles are triangles.***

$$(\underline{\mathrm{C}(\mathcal{A})}(\coprod_{i \in I} \mathrm{A}_i, \mathrm{B}), [\_, \mathrm{d}]) \cong \prod_{i \in I} (\underline{\mathrm{C}(\mathcal{A})}(\mathrm{A}_i, \mathrm{B}), [\_, \mathrm{d}]), \ \mathrm{K}(\mathcal{A}) = \mathrm{H}^0 \underline{\mathrm{C}(\mathcal{A})}.$$

DEFINITION 1.2. A triangulated category is said to have direct sums if it has categorical direct sums, and direct sums of triangles are triangles.

DEFINITION 1.3. Let $\mathscr{S}$ be a triangulated category with arbitrary direct sums. Then a full triangulated subcategory $L \subset \mathscr{S}$ is called *localizing* if

Every direct summand of an object in L is in L. (1.3.1)

Every direct sum of objects of L is in L. (1.3.2)

REMARK 1.4. We will see later that (1.3.1) is superfluous; (1.3.2) $\Rightarrow$ (1.3.1). By

LEMMA 1.5. *If L is a localizing subcategory of the triangulated category, then the triangulated category* $\mathscr{S}/_L$ *has direct sums. In fact, the functor* $\mathscr{S} \to \mathscr{S}/_L$ *preserves direct sums.*

A. Nous avons vu que la somme directe d'une famille quelconque de morphismes surjectifs est surjectif (N° 1); en fait, on voit même que le foncteur $(A_i)_{i \in I} \to \bigoplus_{i \in I} A_i$, défini sur la "catégorie produit" $\mathbf{C}^I$, et à valeurs

dans $\mathbf{C}$, est *exact à droite*. Il est même exact si I est fini, mais pas nécessairement si I est infini, car la somme directe d'une famille infinie de monomorphismes n'est pas nécessairement un monomorphisme, comme nous l'avons remarqué au N°1 (pour la situation duale). D'où l'axiome suivant :

AB 4) *L'axiome* AB 3) *est vérifié, et la somme directe d'une famille de monomorphismes est un monomorphisme.*

AB4) $\mathcal{A}$ satisfies AB3), and the coproduct of a family of monomorphisms is a monomorphism.

Телескоп в триангульованій категорії

EXAMPLE 1.6. Let $L \subset K(\mathscr{A})$ be the subcategory of homologically trivial complexes of objects in the abelian category $\mathscr{A}$. If $\mathscr{A}$ satisfies $AB4$ (i.e. direct sums of exact sequences are exact) then L is localizing.

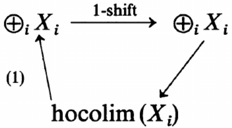
COROLLARY 1.7 *If $\mathscr{A}$ satisfies AB4 then $D(\mathscr{A})=K(\mathscr{A})/_L$ has direct sums.* □

rest of this article, $\mathscr{A}$ will be the category of modules over a ring R, which satisfies both $AB4$ and $AB4^*$.

Let $\mathscr{S}$ be a triangulated category with direct sums. Suppose $\{X_i, i \in \mathbb{N}\}$ is a sequence of objects in $\mathscr{S}$, together with maps $X_i \to X_{i+1}$. We wish to define the homotopy colimit of the sequence.

DEFINITION 2.1. The homotopy colimit of the sequence above is the third edge of the triangle

(1)



where the map (shift) above is the shift map, whose coordinates are the natural maps $X_i \to X_{i+1}$.

REMARK 2.2. This is nothing more than the usual "mapping telescope" construction of topology. If $\mathcal{S} = D(\mathcal{A})$, and $\mathcal{A}$ is an abelian category satisfying *AB*5 (filtered direct limits of exact sequences are exact), the reader will easily prove:

$$H^i\left(\operatorname*{hocolim}_j (X_j)\right) = \operatorname*{colim}_j H^i(X_j). \tag{2.2.1}$$

If we choose actual chain maps of chain complexes $X_i \to X_j$ (not merely homotopy equivalence classes of such maps), then one can prove easily:

$$\text{There is a natural quasi-isomorphism } \operatorname*{hocolim}_i (X_i) \to \operatorname*{colim}_i (X_i). \tag{2.2.2}$$

L'axiome suivant est strictement plus fort que AB 4):

AB 5) *L'axiome* AB 3) *est vérifié, et si* $(A_i)_{i \in I}$ *est une famille filtrante croissante de sous-trucs d'un* $A \in \mathbf{C}$, *B un sous-truc quelconque de* A, *on a* $\left(\sum_i A_i\right) \cap B = \sum_i (A_i \cap B)$.

AB5) $\mathcal{A}$ satisfies AB3), and filtered colimits of exact sequences are exact.

Телескоп відображень в топології

For

$$X_\bullet = \left(X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \right)$$

a sequence in Top, its mapping telescope is the quotient topological space of the disjoint union of product topological spaces

$$\mathrm{Tel}(X_\bullet) \colon \left(\bigsqcup_{n \in \mathbb{N}} (X_n \times [n, n+1]) \right) / \sim$$

where the equivalence relation quotiented out is

$$(x_n, n+1) \sim (f_n(x_n), n+1)$$

for all $n \in \mathbb{N}$ and $x_n \in X_n$.
At least if all the f_n are inclusions, this is the sequential attachment of ever “larger” cylinders, whence the name “telescope”.

Let $\mathcal{S}$ be a triangulated category. Suppose X_i, $i \in \mathbb{N}$, is a sequence of objects in $\mathcal{S}$, together with maps $f_i : X_i \to X_{i+1}$. Then $\forall n \in \mathbb{N}$ there is a split exact sequence in $\mathcal{S}$

$$0 \to \bigoplus_{i=0}^{n} X_i \xrightarrow{1-\text{shift}} \bigoplus_{i=0}^{n+1} X_i \xrightarrow{p} X_{n+1} \to 0,$$

$$1-\text{shift} = \begin{pmatrix} 1 & -f_0 & & & & \\ & 1 & -f_1 & & 0 & \\ & & 1 & -f_2 & & \\ & 0 & & \ddots & \ddots & \\ & & & & 1 & -f_{n-1} \end{pmatrix}, \quad p = \begin{pmatrix} f_0 f_1 \dots f_n \\ f_1 \dots f_n \\ \dots \\ f_{n-1} f_n \\ f_n \\ 1 \end{pmatrix}$$

Splitting is given by

$$\bigoplus_{i=0}^{n} X_i \xleftarrow{q} \bigoplus_{i=0}^{n+1} X_i \xleftarrow{j} X_{n+1},$$

$$q = \begin{pmatrix} 1 & f_0 & f_0 f_1 & & f_0 f_1 \dots f_{n-2} \\ & 1 & f_1 & \ddots & f_1 \dots f_{n-2} \\ & & 1 & \ddots & \dots \\ & 0 & & \ddots & f_{n-2} \\ & & & & 1 \\ & & & & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

The diagram commutes

$$\begin{array}{ccccc}
\bigoplus_{i=0}^{n} X_i & \xrightarrow{1-\text{shift}} & \bigoplus_{i=0}^{n+1} X_i & \xrightarrow{\quad p \quad} & X_{n+1} \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f_{n+1}} \\
\bigoplus_{i=0}^{n+1} X_i & \xrightarrow{1-\text{shift}} & \bigoplus_{i=0}^{n+2} X_i & \xrightarrow{\quad p \quad} & X_{n+2}
\end{array}$$

Let $\mathcal{S} = D(\mathcal{A})$, where abelian category satisfies AB5). Filtered colimit of rows is an exact sequence in $C(\mathcal{A})$

$$0 \to \coprod_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \coprod_{i=0}^{\infty} X_i \xrightarrow{\quad p \quad} \mathsf{colim}_{i \in \mathbb{N}} X_i \to 0.$$

Proposition 1.7.5. *Let $\mathscr{C}$ be an abelian category and let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathbf{C}(\mathscr{C})$. Let $M(f)$ be the mapping cone of f and let $\phi^n : M(f)^n = X^{n+1} \oplus Y^n \to Z^n$ be the morphism $(0, g^n)$. Then $\{\phi^n\}_n : M(f) \to Z$ is a morphism of complexes, $\phi \circ \alpha(f) = g$, and ϕ is a quasi-isomorphism.*

Hence, a quasi-isomorphism

$$\mathsf{hocolim}_i X_i = \mathsf{Cone}(1 - \text{shift}) \to \mathsf{colim}_i X_i.$$

Гомотопійна кограниця

DEFINITION 1.6.4. *Let $\mathcal{T}$ be a triangulated category satisfying [TR5($\aleph_1$)]; that is, countable coproducts exist in $\mathcal{T}$. Let*

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} \cdots$$

be a sequence of objects and morphisms in $\mathcal{T}$. The homotopy colimit *of the sequence, denoted $\underrightarrow{Hocolim}\, X_i$, is by definition given, up to non–canonical isomorphism, by the triangle*

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-shift} \coprod_{i=0}^{\infty} X_i \longrightarrow \underrightarrow{Hocolim}\, X_i \longrightarrow \Sigma\left\{\coprod_{i=0}^{\infty} X_i\right\}$$

where the shift map $\coprod_{i=0}^{\infty} X_i \xrightarrow{shift} \coprod_{i=0}^{\infty} X_i$ is the direct sum of j_{i+1} : $X_i \to X_{i+1}$. In other words, the map $\{1 - shift\}$ is the infinite matrix

$$\begin{pmatrix} 1_{X_0} & 0 & 0 & 0 & \cdots \\ -j_1 & 1_{X_1} & 0 & 0 & \cdots \\ 0 & -j_2 & 1_{X_2} & 0 & \cdots \\ 0 & 0 & -j_3 & 1_{X_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Адитивність гомотопійної кограниці

LEMMA 1.6.5. *If we have two sequences*

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

and

$$Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots$$

then, non–canonically,

$$\underrightarrow{Hocolim}\ \{X_i \oplus Y_i\} = \{\ \underrightarrow{Hocolim}\ X_i\} \oplus \{\ \underrightarrow{Hocolim}\ Y_i\}\,.$$

Proof: Because the direct sum of two triangles is a triangle by Proposition 1.1.20, there is a triangle

$$\left\{\coprod_{i=0}^{\infty} X_i\right\} \oplus \left\{\coprod_{i=0}^{\infty} Y_i\right\} \xrightarrow{1-shift} \left\{\coprod_{i=0}^{\infty} X_i\right\} \oplus \left\{\coprod_{i=0}^{\infty} Y_i\right\}$$

$$\downarrow$$

$$\{\ \underrightarrow{\mathrm{Hocolim}}\ X_i\} \oplus \{\ \underrightarrow{\mathrm{Hocolim}}\ Y_i\}$$

and this triangle identifies

$$\{\ \underrightarrow{\mathrm{Hocolim}}\ X_i\} \oplus \{\ \underrightarrow{\mathrm{Hocolim}}\ Y_i\} \quad = \quad \underrightarrow{\mathrm{Hocolim}}\ \{X_i \oplus Y_i\}\,.$$

□

Гомотопійна кограниця послідовності тотожних морфізмів

Lemma 1.6.6. *Let X be an object of $\mathcal{T}$, and let*

$$X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} \cdots$$

be the sequence where all the maps are identities on X. Then

$$\underrightarrow{Hocolim}\ X = X,$$

even canonically.

Proof: The point is that the map

$$\coprod_{i=0}^{\infty} X \xrightarrow{1 - shift} \coprod_{i=0}^{\infty} X$$

is a split monomorphism.

is split. Perhaps a simpler way to say this is that the map

$$X \oplus \left\{ \coprod_{i=0}^{\infty} X \right\} \xrightarrow{\begin{pmatrix} i_0 & \{1 - shift\} \end{pmatrix}} \coprod_{i=0}^{\infty} X$$

is an isomorphism, where $i_0 : X \longrightarrow \coprod_{i=0}^{\infty} X$ is the inclusion into the zeroth summand. In other words, the candidate triangle

$$\coprod_{i=0}^{\infty} X \xrightarrow{1 - shift} \coprod_{i=0}^{\infty} X \xrightarrow{pr} X \xrightarrow{0} \Sigma \left\{ \coprod_{i=0}^{\infty} X \right\}$$

where $pr : \coprod_{i=0}^{\infty} X \longrightarrow X$ is the map which is 1 on every summand, is isomorphic to the sum of the two triangles

$$\coprod_{i=0}^{\infty} X \xrightarrow{1} \coprod_{i=0}^{\infty} X \longrightarrow 0 \longrightarrow \Sigma \left\{ \coprod_{i=0}^{\infty} X \right\}$$

and

$$0 \longrightarrow X \xrightarrow{1} X \longrightarrow 0.$$

Hence X is identified as $\underrightarrow{\text{Hocolim}}\, X$. □

Another splitting of

$$0 \to \bigoplus_{i=0}^{n} X_i \xrightarrow{1-\text{shift}} \bigoplus_{i=0}^{n+1} X_i \xrightarrow{p} X_{n+1} \to 0,$$

is given by

$$\bigoplus_{i=0}^{n} X_i \xleftarrow{t} \bigoplus_{i=0}^{n+1} X_i \xleftarrow{i_0} X_{n+1},$$

$$t = \begin{pmatrix} 0 & 0 & 0 & & 0 \\ -1 & 0 & 0 & \ddots & 0 \\ -1 & -1 & 0 & \ddots & 0 \\ & \ddots & \ddots & \ddots & \\ -1 & -1 & -1 & \ddots & 0 \\ -1 & -1 & -1 & & -1 \end{pmatrix}, \quad i_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Hence, splitting of

$$0 \to \coprod_{i=0}^{\infty} X \xrightarrow{1-\text{shift}} \coprod_{i=0}^{\infty} X \xrightarrow{p} X \to 0,$$

is given by

$$\coprod_{i=0}^{\infty} X \xleftarrow{t} \coprod_{i=0}^{\infty} X \xleftarrow{i_0} X.$$

The both rows in

$$\begin{array}{ccccccc}
\coprod_{i=0}^{\infty} X & \xrightarrow{\mathsf{in}_2} & X \oplus \coprod_{i=0}^{\infty} X & \xrightarrow{\mathsf{pr}_1} & X & \xrightarrow{0} & \\
{\scriptstyle 1}\downarrow & & \binom{i_0}{1-\text{shift}}\downarrow\cong & & \exists\downarrow\cong & & \\
\coprod_{i=0}^{\infty} X & \xrightarrow{1-\text{shift}} & \coprod_{i=0}^{\infty} X & \longrightarrow & \text{Hocolim}(1) & \longrightarrow &
\end{array}$$

are distinguished.

Гомотопійна кограниця послідовності нульових морфізмів

Lemma 1.6.7. *If in the sequence*

$$X_0 \xrightarrow{\ 0\ } X_1 \xrightarrow{\ 0\ } X_2 \xrightarrow{\ 0\ } \cdots$$

all the maps are zero, then $\underrightarrow{\mathit{Hocolim}}\, X_i = 0$.

Proof: The point is that then the shift map in

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{\ 1 - \mathit{shift}\ } \coprod_{i=0}^{\infty} X_i$$

vanishes. But by [TR0] there is a triangle

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{\ 1\ } \coprod_{i=0}^{\infty} X_i \longrightarrow 0 \longrightarrow \Sigma\left\{\coprod_{i=0}^{\infty} X_i\right\}$$

and this identifies 0 as $\underrightarrow{\mathrm{Hocolim}}\, X_i$. □

Розщеплюваність ідемпотентів

PROPOSITION 1.6.8. *Suppose* $\mathcal{T}$ *is a triangulated category satisfying [TR5(*$\aleph_1$*)]. Let* X *be an object of* $\mathcal{T}$*, and suppose* $e : X \to X$ *is idempotent; that is,* $e^2 = e$*. Then* e *splits in* $\mathcal{T}$*. There are morphisms* f *and* g *below*

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } X$$

with $gf = e$ *and* $fg = 1_Y$*.*

Proof: Cosider the two sequences

$$X \xrightarrow{\ e\ } X \xrightarrow{\ e\ } X \xrightarrow{\ e\ } \cdots$$

and

$$X \xrightarrow{\ 1-e\ } X \xrightarrow{\ 1-e\ } X \xrightarrow{\ 1-e\ } \cdots$$

Let Y be the homotopy colimit of the first, and Z the homotopy colimit of the second. We will denote this by writing $Y = \underrightarrow{\text{Hocolim}}\,(e)$ and $Z = \underrightarrow{\text{Hocolim}}\,(1-e)$.

By Lemma 1.6.5, $Y \oplus Z$ is the homotopy colimit of the direct sum of the two sequences, that is of

$$X \oplus X \xrightarrow{\begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix}} \cdots$$

But the following is a map of sequences

$$\begin{array}{ccccccc}
X \oplus X & \xrightarrow{\begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix}} & X \oplus X & \xrightarrow{\begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix}} & X \oplus X & \xrightarrow{\begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix}} & \cdots \\
{\scriptstyle \begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix}} \Big\downarrow & & {\scriptstyle \begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix}} \Big\downarrow & & {\scriptstyle \begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix}} \Big\downarrow & & \\
X \oplus X & \xrightarrow[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]{} & X \oplus X & \xrightarrow[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]{} & X \oplus X & \xrightarrow[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]{} & \cdots
\end{array}$$

and in fact, the vertical maps are isomorphisms. The map

$$\begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix} : X \oplus X \longrightarrow X \oplus X$$

is its own inverse; its square is easily computed to be the identity.

It follows that the homotopy limits of the two sequences are the same. Thus $Y \oplus Z$ is the homotopy limit of the bottom row, and the bottom row decomposes as the direct sum of the two sequences

$$X \xrightarrow{\ 1\ } X \xrightarrow{\ 1\ } X \xrightarrow{\ 1\ } \cdots$$

and

$$X \xrightarrow{\ 0\ } X \xrightarrow{\ 0\ } X \xrightarrow{\ 0\ } \cdots$$

By Lemma 1.6.6, the homotopy colimit of the first sequence is X, while by Lemma 1.6.7, the homotopy colimit of the second sequence is 0. The homotopy colimit of the sum, which is $Y \oplus Z$, is therefore isomorphic to $X \oplus 0 = X$.

More concretely, consider the maps of sequences

$$\begin{array}{ccccccc} X & \xrightarrow{e} & X & \xrightarrow{e} & X & \xrightarrow{e} & \cdots \\ {\scriptstyle e}\downarrow & & {\scriptstyle e}\downarrow & & {\scriptstyle e}\downarrow & & \\ X & \xrightarrow{1} & X & \xrightarrow{1} & X & \xrightarrow{1} & \cdots \end{array}$$

and

$$\begin{array}{ccccccc} X & \xrightarrow{1-e} & X & \xrightarrow{1-e} & X & \xrightarrow{1-e} & \cdots \\ {\scriptstyle 1-e}\downarrow & & {\scriptstyle 1-e}\downarrow & & {\scriptstyle 1-e}\downarrow & & \\ X & \xrightarrow{1} & X & \xrightarrow{1} & X & \xrightarrow{1} & \cdots \end{array}$$

What we have shown is that the induced maps on homotopy colimits, that is $g : Y \to X$ and $g' : Z \to X$ can be chosen so that the sum $Y \oplus Z \to X$ is an isomorphism.

In the sequence

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots$$

defining Y as the homotopy colimit, we get a map $f : X \to Y$, just the map from a finite term to the colimit. In the sequence

$$X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} \cdots$$

the map from the finite terms to the homotopy colimit is the identity. We deduce a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{1} & X. \end{array}$$

Similarly, from the other sequence we deduce a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f'} & Z \\ {\scriptstyle 1-e}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\ X & \xrightarrow{1} & X. \end{array}$$

In other words, we conclude in total that $e = gf$ and $1 - e = g'f'$. The composite

$$X \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} Y \oplus Z \xrightarrow{\begin{pmatrix} g & g' \end{pmatrix}} X$$

is $e + (1 - e) = 1$. Since we know that the map $Y \oplus Z \to X$ is an isomorphism, it follows that the map $X \to Y \oplus Z$ above is its (two–sided) inverse. The composite in the other order is also the identity. In particular, $fg = 1_Y$ and $f'g' = 1_Z$. □

REMARK 1.6.9. Dually, if $\mathcal{T}$ satisfies [TR5*($\aleph_1$)], then idempotents also split.

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