IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **48** (2015) 365002 (11pp) doi:10.1088/1751-8113/48/36/365002

Thermodynamic properties of the 2+1-dimensional Dirac fermions with broken time-reversal symmetry

S G Sharapov

Bogolyubov Institute for Theoretical Physics, National Academy of Science of Ukraine, 14-b Metrologicheskaya Street, Kiev 03680, Ukraine

E-mail: sharapov@bitp.kiev.ua

Received 4 March 2015, revised 15 July 2015
Accepted for publication 17 July 2015
Published 13 August 2015

Abstract

We study the thermodynamic properties of the two-component 2+1-dimensional massive Dirac fermions in an external magnetic field. The broken time-reversal symmetry results in the presence of a linear in the magnetic field part of the thermodynamic potential, while in the famous problem of Landau diamagnetism the leading field dependent term is quadratic in the field. Accordingly, the leading term of the explicitly calculated magnetization is anomalous, viz it is independent of the strength of the magnetic field. The Středa formula is employed to describe how the anomalous magnetization is related to the anomalous Hall effect.

Keywords: Landau levels, Dirac fermions, magnetic moment

1. Introduction and model

Since 80s of the 20th century the condensed matter realization of the (2 + 1)-dimensional Dirac fermions [1, 2] attracts attention of researchers. The discovery of graphene in 2004 [3] was a tremendous breakthrough in the experimental realization of the massless Dirac fermions. Now a family of the solid-state Dirac materials which includes both the brothers of graphene, viz made in 2010 silicene, announced in 2014 germanene [4], and cousins such as MoS_2, topological insulators, Weyl semimetals, etc. There are even artificially designed in 2012 nephews such as molecular graphene [5] and ultracold fermionic atoms in optical honeycomb lattice [6]. An exciting feature of the latter artificial condensed matter system is that it allows one the experimental realization [7] of the Haldane model [2] where the time-reversal symmetry is broken and a quantum Hall effect appears as an intrinsic property of the band structure in the *absence* of an external magnetic field.

As a matter of fact, the low-energy description of these systems is based on a pair of the independent effective Dirac Hamiltonians

$$\mathcal{H}_\eta(\mathbf{k}) = \hbar v_\mathrm{F}\left(\eta k_x \tau_1 + k_y \tau_2\right) + \Delta_\eta \tau_3, \tag{1}$$

where the Pauli matrices $\boldsymbol{\tau}$ act in the sublattice (pseudospin) space, the two-dimensional wave-vector $\mathbf{k} = (k_x, k_y)$ is counted off the two independent $\mathbf{K}_\eta$ points (with $\eta = \pm$) in the Brillouin zone, and v_F is the Fermi velocity, and Δ_η is the mass (gap) term. Although the spin degree of freedom is neglected for simplicity, it can be easily included if necessary. Accordingly, the most general expression for the gap is given by

$$\Delta_\eta = \Delta + \eta \Delta_T, \tag{2}$$

where the gap Δ is invariant with respect to the time-reversal symmetry, and $\eta\Delta_T$ is not.

In the simplest case [1], $\Delta_T = 0$, the mass term Δ originates from a different on-site energy on two triangular sublattices that form hexagonal lattice. The fermion doubling [8] guarantees that the ordinary solid-state Dirac materials are always described by pairs of the Hamiltonians (1), so that the full Hamiltonian $\mathcal{H}(\mathbf{k}) = \mathcal{H}_{\eta=+1}(\mathbf{k}) \oplus \mathcal{H}_{\eta=-1}(\mathbf{k})$ respects time-reversal symmetry (see [9] for an overview of the discrete symmetries for the various mass terms)

$$\left(\Pi \otimes \tau_0\right)\mathcal{H}^*(\mathbf{k})\left(\Pi \otimes \tau_0\right) = \mathcal{H}(-\mathbf{k}) \tag{3}$$

that involves operator Π swapping $\eta = 1$ and $\eta = -1$ valleys. Here τ_0 is the unit matrix. The inversion symmetry is, however, broken because the sublattices are inequivalent for $\Delta \neq 0$. Obviously, when a separate $\mathbf{K}_\eta$ point is considered, the time-reversal symmetry is always broken.

A more sophisticated case, $\Delta_T \neq 0$, with broken time-reversal symmetry is realized in the Haldane model [2] by including next-nearest-neighbour (nearest-neighbour on the same sublattice) hopping and periodic local magnetic flux density with zero total flux through the unit cell. The experimental implementation of this model in [7] is highly nontrivial. An earlier history of searches of the condensed matter realization of the time-reversal anomaly is presented in [10].

From a theoretical point of view, the specifics of the Haldane model appears to be more transparent when one considers the spectrum of the Hamiltonians (1) in an external magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = (0, 0, B)$ applied perpendicular to the plane along the positive z axis. Accordingly, the momentum operator $\hat{p}_i = -\mathrm{i}\hbar\partial_i$ has to be replaced by the covariant momentum $\hat{p}_i \to \hat{p}_i + \frac{e}{c}A_i$. Here $-e < 0$ is the electron charge and $\mathbf{A} = (0, Bx, 0)$ is the vector potential in the Landau gauge. The corresponding Landau level energies are

$$\epsilon_{n\eta} = \begin{cases} -\eta\Delta_\eta \,\mathrm{sgn}(eB), & n = 0, \\ \pm M_n, & n = 1, 2, \ldots, \end{cases} \tag{4}$$

where $M_n = \sqrt{\Delta_\eta^2 + 2nv_F^2\hbar\,|eB|/c}$.

In 2+1-dimensions there are two inequivalent irreducible 2×2 representations of the Dirac algebra labelled by $a = 1{,}2$. In particular, the Hamiltonian (1) corresponds to the representation $\gamma_a^\nu = (\tau_3, -\mathrm{i}\eta\tau_2, \mathrm{i}\tau_1)$ with $\nu = 0, 1, 2$. In general, one can relate the sign of the energy of the $n = 0$ Landau level to the sign of the product of the mass, Δ_η, and the signature, η_a, of the Dirac matrices γ_a^ν. The signature is defined as [11]

$$\eta_a = \frac{\mathrm{i}}{2}\,\mathrm{tr}\left[\gamma_a^0\gamma_a^1\gamma_a^2\right] = \pm 1 \tag{5}$$

and is called sometimes 'chirality'. Thus, instead of putting the $\pm$ sign in the matrix γ_a^1, one can instead include this sign in the mass term (see e.g. [1, 12]).

As mentioned before, in the time-reversal symmetric case, $\Delta_T = 0$, the spectrum of the full Hamiltonian $\mathcal{H}(\mathbf{k}) = \mathcal{H}_{\eta=+1}(\mathbf{k}) \oplus \mathcal{H}_{\eta=-1}(\mathbf{k})$ is particle-hole symmetric and invariant under $B \to -B$. The case $\Delta_T \neq 0$ and $\Delta = 0$ corresponds to the so called Haldane mass that breaks time-reversal symmetry. As the result, the $n = 0$ 'zero-mode' level breaks both the particle-hole symmetry and invariance under $B \to -B$. It is crucial that the difference between these cases survives even in the $B = 0$ limit, making possible the quantum Hall effect in the absence of magnetic field.

The vast majority of the literature (see e.g. [13–15] and references therein) on the thermodynamic properties of the Dirac fermions including magnetic oscillations is devoted to the case of the particle-hole symmetric spectrum. This concerns both the papers that consider the reducible 4×4 representation of the Dirac algebra in 2+1-dimensions and present a field theoretical perspective of the problem [13] and the works devoted to the condensed matter systems [14, 15] with even number of fermion species. Since the time-reversal symmetry is preserved, a finite Hall conductivity is only possible in nonzero external magnetic field. Respectively, the first magnetic field dependent term of the grand thermodynamic potential $\Omega(\mu, B)$ is proportional to B^2, so that there is no net magnetization in the limit $B \to 0$.

The purpose of the present work is to study the thermodynamic properties of the Dirac fermions with the broken time-reversal symmetry. In other words, we focus on the asymmetric spectrum (4) associated with the Hamiltonian (1) for one fixed value of η, viz we take $\Delta_T = 0$, and have $\Delta_\eta = \Delta$. Having the thermodynamic potential for 'unphysical' case with a fixed η, it is straightforward to obtain, for example, the result for two $\mathbf{K}_\pm$ points with the Haldane mass and include the spin degree of freedom.

The paper is organized as follows. We already began by presenting in section 1 the problem and the model Hamiltonian (1). In section 2 we consider the grand thermodynamic potential. We show that the unboundedness of the Landau level spectrum for the negative energies results in the diverging carrier density and discuss the relativistic form of the thermodynamic potential that corresponds to the finite carrier imbalance. The magnetic field dependent parts of the carrier imbalance and density are obtained in section 3 using the Euler–Maclaurin formula. The main results of the present paper that include thermodynamic potential and magnetization in the low magnetic field regime are presented in section 4. Using the Středa formula we overview in section 5 how the obtained expressions can be linked to the known results for the Hall conductivity in the anomalous and normal cases. In section 6, the main results of the paper are summarized.

2. Grand thermodynamic potential and its specific in the Dirac case

The grand thermodynamic potential [16] can be written as follows

$$\Omega(T, \mu) = -T \int_{-\infty}^{\infty} \mathrm{d}\epsilon D(\epsilon) \ln \left(1 + \mathrm{e}^{\frac{\mu-\epsilon}{T}} \right), \tag{6}$$

where T is the temperature, μ is the chemical potential, and $D(\epsilon)$ is the density of states (DOSs). We have also set the Boltzmann constant $k_{\mathrm{B}} = 1$. The derivative of the thermodynamic potential (6) with respect to the chemical potential μ,

$$\rho(T, \mu) = -\frac{1}{V} \frac{\partial \Omega(T, \mu)}{\partial \mu} = \int_{-\infty}^{\infty} \mathrm{d}\epsilon D(\epsilon) n_{\mathrm{F}}(\epsilon), \tag{7}$$

determines the density of carriers ρ in nonrelativistic many-body theory as a function of T, B, and μ. Here $n_{\mathrm{F}}(\epsilon) = 1/[\exp[(\epsilon - \mu)/T] + 1]$ is the usual Fermi function and V is volume (area) of the system which is set to be unit.

In the absence of scattering from impurities the DOS per unit area for a given value of η is expressed via the energies of the Landau levels $\epsilon_{n\eta}$ as follows

$$D_0(\epsilon) = \frac{|eB|}{2\pi\hbar c}\left[\delta(\epsilon + \eta\Delta\,\mathrm{sgn}(eB)) + \sum_{n=1}^{\infty}\left[\delta\left(\epsilon - M_n\right) + \delta\left(\epsilon + M_n\right)\right]\right]. \tag{8}$$

Substituting the DOS $D_0(\epsilon)$ given by equation (8) in equation (7) we obtain

$$\rho \equiv \rho_0 + \rho_{n\geqslant 1} = \frac{|eB|}{2\pi\hbar c}\left[n_{\mathrm{F}}(-\eta\Delta\mathrm{sign}(eB)) + \sum_{n=1}^{\infty}\left[n_{\mathrm{F}}\left(M_n\right) + n_{\mathrm{F}}\left(-M_n\right)\right]\right], \tag{9}$$

where ρ_0 and $\rho_{n\geqslant 1}$ denote contributions from the lowest, $n = 0$, and remaining ($\pm M_n$, $n \neq 0$) Landau levels, respectively. We observe that equation (9) diverges because the spectrum (4) is unbound for the negative energies $\epsilon_{n\eta}$. Since the Dirac Hamiltonian (1) has to be regarded as an effective model Hamiltonian derived from the tight-binding model, the divergence can be removed by using the appropriate band-width cutoff.

The other way to remove this divergency is to begin with so called relativistic thermodynamic potential (see [13, 17, 18])

$$\Omega^{\mathrm{rel}}(T,\,\mu) = -T\int_{-\infty}^{\infty}\mathrm{d}\epsilon D(\epsilon)\ln\left(2\cosh\frac{\epsilon-\mu}{2T}\right). \tag{10}$$

To clarify the physical meaning of the potential (10) we differentiate it with respect to μ and obtain the relativistic carrier density

$$\rho^{\mathrm{rel}}(T,\,\mu) = -\frac{\partial\Omega^{\mathrm{rel}}(T,\,\mu)}{\partial\mu} = -\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\epsilon D(\epsilon)\tanh\frac{\epsilon-\mu}{2T}. \tag{11}$$

Then for the particle-hole symmetric spectrum, i.e. when the DOS $D(\epsilon)$ is an even function of the energy ϵ, $D(\epsilon) = D(-\epsilon)$ one can see that $\rho^{\mathrm{rel}}(T,\,\mu = 0) = 0$. Further, using the identity $1 = \theta(\epsilon) + \theta(-\epsilon)$ with $\theta(\epsilon)$ being a step function, one can rewrite the last equation in the following form

$$\rho^{\mathrm{rel}} = \int_{-\infty}^{\infty}\mathrm{d}\epsilon D(\epsilon)\left[n_{\mathrm{F}}(\epsilon)\theta(\epsilon) - \left[1 - n_{\mathrm{F}}(\epsilon)\right]\theta(-\epsilon)\right]. \tag{12}$$

This shows that ρ^{rel} corresponds to the relativistic carrier density or the carrier imbalance, viz $\rho^{\mathrm{rel}} = \rho_+ - \rho_-$, where ρ_+ and ρ_- are the densities of 'electrons' and 'holes', respectively. Accordingly, when the DOS is the even function of energy, the relativistic thermodynamic potential (10) can be presented as a sum of the vacuum, electron and hole terms [13] illustrating its physical meaning. The interpretation of ρ^{rel} for the case of the asymmetric DOS is discussed in detail in [17] (see also [19] for a review).

Substituting the DOS, $D_0(\epsilon)$ given by equation (8) in equation (11), one obtains

$$\begin{aligned}\rho^{\mathrm{rel}} \equiv \rho_0^{\mathrm{rel}} + \rho_{n\geqslant 1}^{\mathrm{rel}} = \frac{|eB|}{4\pi\hbar c}\Bigg[&\tanh\frac{\mu + \eta\Delta\mathrm{sign}(eB)}{2T} \\ &+ \sum_{n=1}^{\infty}\left[\tanh\frac{\mu - M_n}{2T} + \tanh\frac{\mu + M_n}{2T}\right]\Bigg],\end{aligned} \tag{13}$$

where ρ_0^{rel} and $\rho_{n\geqslant 1}^{\mathrm{rel}}$ denote contributions from the lowest and remaining Landau levels, respectively. One can see that, in contrast to equation (9), equation (13) converges. Thus the use of the relativistic potential (10) makes the calculation free of divergencies, but at the end one should prove that the physical results do not change as compared to the results that follow directly from the initial potential (6).

In general, one can relate the grand thermodynamic potential (6) and the relativistic potential (10) as follows

$$\Omega(T,\,\mu) = \Omega^{\rm rel}(T,\,\mu) - \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\epsilon\,(\mu - \epsilon) D(\epsilon). \tag{14}$$

When the $D(\epsilon)$ is an even function of the energy ϵ, the last term with the integral in equation (14) reduces to

$$-\mu\rho_{\rm hf} = -\mu\int_{-\infty}^{\infty} \mathrm{d}\epsilon D(\epsilon)\theta(-\epsilon). \tag{15}$$

Then the carrier imbalance $\rho^{\rm rel}$ and total carrier density ρ are related by the expression $\rho^{\rm rel}(T,\,\mu) = \rho(T,\,\mu) - \rho_{\rm hf}$, where $\rho_{\rm hf}$ is the density of particles for a half-filled band (in the lower Dirac cone). Note that both $\rho(T,\,\mu)$ and $\rho_{\rm hf}$ are divergent, so that the appropriate cutoff has to be introduced. This reflects the already mentioned fact that in the continuum field theory there is no lower bound to the Dirac sea of filled electron states.

The thermodynamic potential (10) can be restored from the particle density $\rho^{\rm rel}(\mu,\,T,\,B)$ by integration

$$\Omega^{\rm rel}(T,\,\mu,\,B) = -\int_{-\infty}^{\mu} \mathrm{d}\epsilon\rho^{\rm rel}(T,\,\epsilon,\,B) + \Omega_c(T,\,B). \tag{16}$$

Here $\Omega_c(T,\,B)$ is a constant of integration. Similarly, integrating $\rho(\mu,\,T,\,B)$ one restores the potential (6).

The magnetization M in the direction perpendicular to the plane is defined in the grand canonical ensemble by the derivative with respect to B at fixed chemical potential, i.e.

$$M(T,\,\mu,\,B) = -\frac{\partial\Omega(T,\,\mu,\,B)}{\partial B}. \tag{17}$$

Thus one needs only the magnetic field dependent part of the grand thermodynamic potentials (10) and (6). A special interest for us represents a linear in magnetic field part of the thermodynamic potential those presence would imply a nonzero net magnetization in the limit of $B \to 0$.

3. Calculation of the carrier imbalance and density

Since we are interested in the weak field regime, there is no need to evaluate the sum over Landau levels exactly. The simplest way to extract the magnetic field dependent terms is to use the Euler–Maclaurin formula

$$\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) \approx \int_0^{\infty} F(x)\mathrm{d}x - \frac{1}{12}F'(0) \tag{18}$$

following the seminal paper on Landau diamagnetism [20]. Firstly we calculate the carrier imbalance $\rho^{\rm rel}_{n\geqslant 1}$ from equation (13), so that

$$\rho^{\rm rel}_{n\geqslant 1} = \frac{|eB|}{4\pi\hbar c}\sum_{n=1}^{\infty} F^{\rm rel}(n) \tag{19}$$

with

$$F^{\text{rel}}(n) = \tanh \frac{\mu - M_n}{2T} + \tanh \frac{\mu + M_n}{2T}. \tag{20}$$

Repeating the arguments of [16, 20] one can see that the use of the Euler–Maclaurin formula is justified in the weak-field regime, $L(B) \ll |\mu|$ with $L(B) = \sqrt{2|eB|\hbar v_{\text{F}}^2/c}$ being the Landau scale.

One can see that the term with the integral in equation (18) is independent of the magnetic field similarly to the original Landau consideration [20]. Combining together the lowest Landau contribution, ρ_0^{rel} and term containing $(-1/2)F^{\text{rel}}(0)$, we obtain the linear in B contribution, $\rho_{\text{I}}^{\text{rel}}$, to ρ^{rel}:

$$\rho_{\text{I}}^{\text{rel}} = \frac{|eB|}{4\pi\hbar c}\left[\tanh \frac{\mu + \eta\Delta \text{sign}(eB)}{2T} - \frac{1}{2}\tanh\frac{\mu - |\Delta|}{2T} - \frac{1}{2}\tanh\frac{\mu + |\Delta|}{2T}\right]. \tag{21}$$

Analyzing all possible cases eB, $\eta\Delta \gtrless 0$, one can simplify the last expression to the form

$$\rho_{\text{I}}^{\text{rel}} = \frac{eB\, \text{sgn}(\eta\Delta)}{8\pi\hbar c}\left[\tanh\frac{\mu + |\Delta|}{2T} - \tanh\frac{\mu - |\Delta|}{2T}\right]. \tag{22}$$

Note that there is no linear in B term in the analysis of Landau diamagnetism [16, 20]. This term would also be absent if one considers the total contribution from both valleys with $\eta = \pm$. Similarly to [16, 20], the term with the derivative $(F^{\text{rel}})'(0)$ produces the quadratic in B part, $\rho_{\text{II}}^{\text{rel}}$, of ρ^{rel}:

$$\rho_{\text{II}}^{\text{rel}} = -\frac{(eB)^2 v_{\text{F}}^2}{96\pi c^2 |\Delta| T}\left[\frac{1}{\cosh^2 \frac{\mu + |\Delta|}{2T}} - \frac{1}{\cosh^2 \frac{\mu - |\Delta|}{2T}}\right]. \tag{23}$$

The same approach can be applied to the carrier density (9). However, as mentioned earlier, the ‘nonrelativistic’ analog of the sum (19) with

$$F(n) = 2\left[n_{\text{F}}(M_n) + n_{\text{F}}(-M_n)\right] \tag{24}$$

is diverging. To make the series convergent, one can introduce the regularization factor, $\exp(-\delta M_n)$ with $\delta \to +0$. Then the use of the Euler–Maclaurin formula (18) is justified. Taking into account the identity $n_{\text{F}}(\epsilon) = 1/2[1 + \tanh(\mu - \epsilon)/2T]$, one can see that the field dependent densities coincide, viz $\rho_{\text{I}} = \rho_{\text{I}}^{\text{rel}}$ and $\rho_{\text{II}} = \rho_{\text{II}}^{\text{rel}}$.

One can also come to the same conclusion that the magnetic field dependent parts of the carrier densities ρ and ρ^{rel} coincide by using equation (14). Indeed, taking the derivative over μ, one can see that

$$\rho^{\text{rel}}(T, \mu, B) = \rho(T, \mu, B) - \frac{1}{2}\int_{-\infty}^{\infty} \text{d}\epsilon D(\epsilon). \tag{25}$$

Here, in contrast to equation (15) we do not assume the evenness of the DOS $D(\epsilon)$. The integral in the rhs of equation (25) corresponds to the total number of states in our bands which should be independent of the magnetic field [21].

4. Magnetic field dependent part of the grand thermodynamic potential and magnetization

Using equation (16) we restore the thermodynamic potentials, viz

$$\Omega^{\text{rel}}(T,\,\mu,\,B) = \Omega^{\text{rel}}(T,\,\mu,\,B=0) + \Omega_{\text{I}}^{\text{rel}}(T,\,\mu,\,B) + \Omega_{\text{II}}^{\text{rel}}(T,\,\mu,\,B) + \Omega_c(T,\,B), \tag{26}$$

and

$$\Omega(T,\,\mu,\,B) = \Omega^{\text{rel}}(T,\,\mu,\,B) + \tilde{\Omega}_c(T,\,B). \tag{27}$$

Here $\Omega_{\text{I}}^{\text{rel}}$ and $\Omega_{\text{II}}^{\text{rel}}$ are, respectively, the linear and quadratic in B parts of the thermodynamic potential obtained by integration of equation (22) for $\rho_{\text{I}}^{\text{rel}}$ and equation (23) for $\rho_{\text{II}}^{\text{rel}}$ from $-\infty$ to μ:

$$\Omega_{\text{I}}^{\text{rel}}(T,\,\mu,\,B) = -\frac{eB\,\text{sgn}(\eta\Delta)T}{4\pi\hbar c}\left[\ln\cosh\frac{\mu+|\Delta|}{2T} - \ln\cosh\frac{\mu-|\Delta|}{2T}\right] \tag{28}$$

and

$$\Omega_{\text{II}}^{\text{rel}}(T,\,\mu,\,B) = \frac{(eB)^2 v_{\text{F}}^2}{48\pi c^2|\Delta|}\left[\tanh\frac{\mu+|\Delta|}{2T} - \tanh\frac{\mu-|\Delta|}{2T}\right]. \tag{29}$$

In equations (26) and (27) the arbitrary functions $\Omega_c(T,\,B)$ and $\tilde{\Omega}_c(T,\,B)$ may depend on T and/or B, but they are independent of μ.

Then the magnetization M defined by equation (17) in the low-field limit reads

$$M(T,\,\mu,\,B) = M_{\text{I}}(T,\,\mu) + M_{\text{II}}(T,\,\mu,\,B) + M_c(T,\,B), \tag{30}$$

where

$$M_{\text{I}} = -\frac{\partial\Omega_{\text{I}}^{\text{rel}}}{\partial B} = \frac{e\,\text{sgn}(\eta\Delta)T}{4\pi\hbar c}\left[\ln\cosh\frac{\mu+|\Delta|}{2T} - \ln\cosh\frac{\mu-|\Delta|}{2T}\right], \tag{31}$$

is the anomalous magnetization,

$$M_{\text{II}} = -\frac{\partial\Omega_{\text{II}}^{\text{rel}}}{\partial B} = -\frac{e}{24\pi\hbar c}\frac{eB\hbar v_{\text{F}}^2}{c|\Delta|}\left[\tanh\frac{\mu+|\Delta|}{2T} - \tanh\frac{\mu-|\Delta|}{2T}\right], \tag{32}$$

is the linear in field part, and the function $M_c(T,\,B)$ is independent of μ. Our final result (30) with M_{I} and M_{II} given by equations (31) and (32) does not depend on a choice of the starting thermodynamic potential that may be taken either (10) or (6). The condition that at half-filling the anomalous magnetization is absent, viz $M(T,\,\mu=0,\,B=0)=0$ allows to fix the function $M_c(T,\,B)=0$.

Persisting in the $B\to 0$ limit anomalous magnetization M_{I} was derived in [22] using the low-field expansion for the Green's function in an external magnetic field. This powerful and more complicated method allows one to consider interacting systems, while the presented here method uses the explicit form of the noninteracting spectrum (4). In the $T=0$ limit equation (31) acquires the form

$$M_{\mathrm{I}} = \frac{e}{4\pi\hbar c}[\eta\Delta\,\mathrm{sgn}\,(\mu)\theta(|\mu| - |\Delta|) + \mu\,\mathrm{sgn}\,(\eta\Delta)\theta(|\Delta| - |\mu|)]. \tag{33}$$

For $\Delta_T = 0$ and $\Delta_\eta = \Delta$, the anomalous M_{I} contribution from $\mathbf{K}_\pm$ points has the opposite sign. Accordingly, when the integral contribution of both valleys is considered, the leading contribution to the magnetization is linear in B term given by equation (32). In the $T = 0$ limit it reduces to

$$M_{\mathrm{II}} = -\frac{e}{12\pi\hbar c}\frac{eB\hbar v_{\mathrm{F}}^2}{c|\Delta|}\theta\left(\Delta^2 - \mu^2\right), \quad L(B) \ll |\Delta|. \tag{34}$$

This result is in agreement with the papers [14, 23] where the gapped graphene was considered, if one takes into account the factor of 4 from the spin-valley degeneracy. In the limit $\Delta \to 0$ the corresponding magnetic susceptibility was firstly considered by McClure [24] in the framework of the studies of diamagnetism of graphite. In the opposite limit, $L(B) \gg |\Delta|$, the vacuum term results in the magnetization $M(\mu = T = \Delta = 0) \sim -\sqrt{B}$ (see [14, 25–27] and references therein). Finally we note that the impact of disorder on the magnetization and susceptibility of graphene with $\Delta = 0$ was studied in [28].

5. Hall conductivity

It is shown in [22] that the anomalous magnetization (31) is crucial in obtaining the off-diagonal thermal transport coefficient. Below we illustrate how this magnetization can be used to obtain the anomalous Hall conductivity and show how to obtain the Hall conductivity quantization in graphene from equation (13).

5.1. Anomalous quantum Hall effect

When the chemical potential μ falls in a gap of the energy spectrum, the Hall conductivity can be found from the Středa formula [29]:

$$\sigma_{xy} = -ec\left(\frac{\partial\rho}{\partial B}\right)_\mu = -ec\left(\frac{\partial M}{\partial\mu}\right)_{\mathrm{B}}, \tag{35}$$

where the second equality follows from the Maxwell relation. Then either substituting the carrier density (22) in the first equality of equation (35) or substituting the anomalous magnetization (31) in the second equality of equation (35), one obtains

$$\sigma_{xy}(B = 0) = -\frac{e^2\mathrm{sign}\,(\eta\Delta)}{8\pi\hbar}\left[\tanh\frac{\mu + |\Delta|}{2T} - \tanh\frac{\mu - |\Delta|}{2T}\right]. \tag{36}$$

We stress that this value of the anomalous Hall conductivity does not depend on a choice of the starting thermodynamic potential. Furthermore, because equation (36) corresponds to the second derivative of the thermodynamic potential with respect to both B and μ, the presence of an arbitrary function of T and B in Ω and in the magnetization (30) does not affect the final result.

In the limit $T = 0$ the last equation reduces to

$$\sigma_{xy} = -\frac{e^2}{4\pi\hbar}\mathrm{sign}\,(\eta\Delta)\theta(|\Delta| - |\mu|). \tag{37}$$

It is interesting to note that this result corresponds to the long-wavelength limit of the static Hall conductivity, viz $\lim_{q\to 0}\lim_{\omega\to 0}\sigma_{xy}(\omega, q)$ (see [30] and the discussion in [31]). The usual Hall conductivity, relevant for the transport, is given by an opposite limit,

$\sigma_{xy}^{\mathrm{tr}} = \lim_{\omega\to 0}\lim_{q\to 0}\sigma_{xy}(\omega, q)$. This anomalous Hall conductivity was studied in [32, 33] and in the clean limit for $\eta = +$ reads

$$\sigma_{xy}^{\mathrm{tr}} = -\frac{e^2\,\mathrm{sgn}\,(\Delta)}{4\pi\hbar}\begin{cases}1, & |\mu| \leqslant |\Delta|,\\ |\Delta|/|\mu|, & |\mu| > |\Delta|.\end{cases} \tag{38}$$

We observe that equation (38) agrees with equation (37) when the chemical potential is inside the gap, $|\mu| < |\Delta|$ in accord with the conditions of the Stȓeda formula [29] validity.

Finally we note that the presence of disorder might regularize [30] the uncertainty of the order of limits mentioned below equation (37). For $|\mu| \leqslant |\Delta|$ the quantized value of the Hall conductivity (38) is robust with respect to disorder, while for $|\mu| > |\Delta|$ the presence of disorder modifies the result for the clean case (see [32, 33] and [34] for a recent discussion).

5.2. Quantum Hall effect in graphene

So far we have considered the anomalous Hall effect by taking the limit $B = 0$. To complete our consideration is instructive to consider the Hall conductivity at finite field. Since the magnetization (30) is valid in the low-field regime, we come back to equation (13) for the carrier density, because it is valid for an arbitrary strength of the magnetic field. The same expression can be directly obtained from the Green's function by doing the summation over Matsubara frequencies [17, 35, 36]. It is important that, in contrast to the the nonrelativistic case, the Matsubara summation in the relativistic case is done [36] without an additional convergence factor. When this factor is present, the Matsubara sum produces the Fermi function n_{F} as we had in equation (9), rather than tanh that is present in equation (13). Note also that to compare the corresponding to equation (13) expression from [36], one should rewrite the lowest Landau level contribution ρ_0^{rel} as follows

$$\rho_0^{\mathrm{rel}} = \frac{eB}{4\pi\hbar c}\left[\theta(eB)\tanh\frac{\mu+\eta\Delta}{2T} - \theta(-eB)\tanh\frac{\mu-\eta\Delta}{2T}\right], \tag{39}$$

where we used that $|eB| = eB\,\mathrm{sgn}(eB) = eB[\theta(eB) - \theta(-eB)]$. Using that in the $T \to 0$ limit $\tanh(\epsilon - \mu)/T \to 1 - 2\theta(-\epsilon + \mu)$, one can show that equation (39) at $T = 0$ acquires the form

$$\rho_0^{\mathrm{rel}} = \frac{eB}{4\pi\hbar c}\,\mathrm{sgn}\,(\eta\Delta)\theta(|\Delta| - |\mu|) + \frac{|eB|}{4\pi\hbar c}\,\mathrm{sgn}\,\mu\theta(|\mu| - |\Delta|). \tag{40}$$

Similarly one can consider the $T = 0$ limit for $\rho_{n\geqslant 1}^{\mathrm{rel}}$ part given by equation (19) and obtain

$$\rho_{n\geqslant 1} = \frac{|eB|\mathrm{sgn}\,(\mu)}{2\pi\hbar c}\sum_{n=1}^{\infty}\theta\big(|\mu| - M_n\big). \tag{41}$$

One can see that the first term of equation (40) corresponding to the anomalous Hall conductivity (36) is time-reversal symmetry breaking, since it is odd under $B \to -B$. On the other hand, the second term of equation (40) and the whole equation (41) are even under $B \to -B$. For $\mu = 0$ in the limit $\Delta \to 0$ the first term of equation (40) leaves behind the $\mathrm{sgn}(\eta\Delta)$. This property is called a sign anomaly. As noted in [17], this anomaly is removed by a finite density effects. Indeed, for $\mu \neq 0$ only the second term of equation (40) survives in the limit $\Delta \to 0$.

Adding together ρ_0^{rel} and $\rho_{n\geqslant 1}^{\mathrm{rel}}$ we recover the results of Lykken *et al* [35] and Schakel [36]

$$\rho^{\mathrm{rel}} = \frac{eB}{4\pi\hbar c}\operatorname{sgn}(\eta\Delta)\theta(|\Delta| - |\mu|) + \frac{|eB|\operatorname{sgn}(\mu)}{2\pi\hbar c}\left(\frac{1}{2} + \left[\frac{(\mu^2 - \Delta^2)c}{2\hbar v_{\mathrm{F}}^2|eB|}\right]\right), \tag{42}$$

where $[x]$ denotes the integer part of x. As discussed above, graphene corresponds to the case with $\Delta_T = 0$, $\Delta_\eta = \Delta \to 0$. Taking into account the spin-valley degeneracy and using the Středa formula (35) we reproduce the half-integer quantum Hall effect [37–39]

$$\sigma_{xy} = -\frac{2e^2\operatorname{sgn}(eB)\operatorname{sgn}(\mu)}{\pi\hbar}\left(\frac{1}{2} + \left[\frac{\mu^2 c}{2\hbar v_{\mathrm{F}}^2|eB|}\right]\right). \tag{43}$$

The argument given below equation (25) allows to conclude that the same result also follows from the nonrelativistic carrier density. The Hall conductivity (43) can also be written as a function of the free carrier imbalance, $\rho_{\mathrm{free}}^{\mathrm{rel}}$,

$$\sigma_{xy} = -\frac{2e^2\operatorname{sgn}(eB)\operatorname{sgn}(\mu)}{\pi\hbar}\left(\frac{1}{2} + \left[\frac{\pi\hbar c\rho_{\mathrm{free}}^{\mathrm{rel}}}{2|eB|}\right]\right), \tag{44}$$

where we used that in the absence of magnetic field $\rho_{\mathrm{free}}^{\mathrm{rel}} = \mu^2\operatorname{sgn}(\mu)/(\pi\hbar^2 v_{\mathrm{F}}^2)$. Concluding it is worth to stress that the quantum Hall effect is a disorder-induced phenomenon. Accordingly the presented consideration only illustrates the expected quantization of the Hall conductivity, rather than explains it.

6. Conclusion

In the present work we demonstrated how the Landau approach used to describe the diamagnetism of electron gas can be applied for the 2+1-dimensional massive Dirac fermions. We derived an explicit expression (30) for magnetization in the low-field regime. It contains field independent (anomalous) term (31) and the linear in field part (32). While there exist many approaches that allow one to obtain these terms, the use of the Euler–Maclaurin formula as in the original work Landau [20] is the simplest option. Using the Středa formula it is illustrated how the obtained expressions for the carrier imbalance (22) (or equation (31) for the magnetization) and (42) are related to the anomalous quantum Hall effect and quantum Hall effect in graphene.

Acknowledgments

SG Sh gratefully acknowledges EV Gorbar, VP Gusynin, VYu Tsaran, and AA Varlamov for helpful discussions.

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