12. Гомотопійне відтягування. Навколо похідних категорій

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Власні справа модельні категорії

The model category $\mathcal{M}$ will be called *right proper* if every pullback of a weak equivalence along a fibration (see Definition 7.2.10) is a weak equivalence.

Every pullback of a weak equivalence between fibrant objects along a fibration (see Definition 7.2.10) is a weak equivalence.

13.3.1. Homotopy pullbacks. If $\mathcal{M}$ is a right proper model category (see Definition 13.1.1), then the homotopy pullback of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ is constructed by replacing g and h by fibrations and then taking a pullback (see Definition 13.3.2). In order to have a well defined functor, we need to choose a fixed functor to convert our maps into fibrations. We will show, however, that any other factorization into a weak equivalence followed by a fibration yields an object naturally weakly equivalent to the homotopy pullback and that, in fact, only one of the maps must be converted to a fibration (see Proposition 13.3.7). Thus, if either of the maps is already a fibration, then the pullback is naturally weakly equivalent to the homotopy pullback (see Corollary 13.3.8).

DEFINITION 13.3.2. Let $\mathcal{M}$ be a right proper model category and let E be an arbitrary but fixed functorial factorization of every map $g\colon X \to Y$ into $X \xrightarrow{i_g} \mathrm{E}(g) \xrightarrow{p_g} Y$, where i_g is a trivial cofibration and p_g is a fibration. The *homotopy pullback* of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ is defined to be the pullback of the diagram $\mathrm{E}(g) \xrightarrow{p_g} Z \xleftarrow{p_h} \mathrm{E}(h)$.

LEMMA 13.3.3. *Let $\mathcal{M}$ be a right proper model category. If $g\colon X \to Y$ is a weak equivalence and $h\colon W \to Z$ is a fibration, then, for any map $k\colon Y \to Z$, the natural map from the pullback of the diagram $X \xrightarrow{kg} Z \xleftarrow{h} W$ to the pullback of the diagram $Y \xrightarrow{k} Z \xleftarrow{h} W$ is a weak equivalence.*

PROOF. We have the commutative diagram

$$\begin{array}{ccccc} X \times_Z W & \longrightarrow & Y \times_Z W & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[g]{} & Y & \xrightarrow[k]{} & Z \end{array}$$

in which the vertical maps are all fibrations. Since g is a weak equivalence, the result follows from Proposition 7.2.14. □

Гомотопійна інваріантність гомотопійного відтягування

PROPOSITION 13.3.4 (Homotopy invariance of the homotopy pullback). *Let $\mathcal{M}$ be a right proper model category. If we have the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{g} & Z & \xleftarrow{h} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{X} & \xrightarrow[\tilde{g}]{} & \widetilde{Z} & \xleftarrow[\tilde{h}]{} & \widetilde{Y} \end{array}$$

in which the vertical maps are weak equivalences, then the induced map of homotopy pullbacks

$$\mathrm{E}(g) \times_Z \mathrm{E}(h) \to \mathrm{E}(\tilde{g}) \times_{\widetilde{Z}} \mathrm{E}(\tilde{h})$$

is a weak equivalence.

PROOF. It is sufficient to show that if g, h, $\tilde{g}$, and $\tilde{h}$ are fibrations, then the map of pullbacks $X \times_Z Y \to \widetilde{X} \times_{\widetilde{Z}} \widetilde{Y}$ is a weak equivalence. This map equals the composition

$$X \times_Z Y \to (\widetilde{X} \times_{\widetilde{Z}} Z) \times_Z Y \approx \widetilde{X} \times_{\widetilde{Z}} Y \to \widetilde{X} \times_{\widetilde{Z}} \widetilde{Y}.$$

Since $\mathcal{M}$ is a right proper model category, the map $X \to \widetilde{X} \times_{\widetilde{Z}} Z$ is a weak equivalence, and Lemma 13.3.3 implies that the last map in the composition is a weak equivalence. $\square$

Три формули для гомотопійного відтягування

Proposition 13.3.7. *Let $\mathcal{M}$ be a right proper model category. If $X \xrightarrow{j_g} W_g \xrightarrow{q_g} Z$ and $Y \xrightarrow{j_h} W_h \xrightarrow{q_h} Z$ are factorizations of, respectively, $g\colon X \to Z$ and $h\colon Y \to Z$, j_g and j_h are weak equivalences, and q_g and q_h are fibrations, then the homotopy pullback of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ is naturally weakly equivalent to each of $W_g \times_Z W_h$, $W_g \times_Z Y$, and $X \times_Z W_h$.*

Proof. If E is the natural factorization used in Definition 13.3.2, then Lemma 13.3.3 implies that the homotopy pullback $\mathrm{E}(g) \times_Z \mathrm{E}(h)$ is naturally weakly equivalent to both $\mathrm{E}(g) \times_Z Y$ and $X \times_Z \mathrm{E}(h)$. Lemma 13.3.3 implies that these are naturally weakly equivalent to $\mathrm{E}(g) \times_Z W_h$ and $W_g \times_Z \mathrm{E}(h)$ respectively, and that these are naturally weakly equivalent to $X \times_Z W_h$ and $W_g \times_Z Y$, respectively. Lemma 13.3.3 implies that both of these are naturally weakly equivalent to $W_g \times_Z W_h$.

□

Corollary 13.3.8. *Let $\mathcal{M}$ be a right proper model category. If at least one of the maps $g\colon X \to Z$ and $h\colon Y \to Z$ is a fibration, then the pullback $X \times_Z Y$ is naturally weakly equivalent to the homotopy pullback of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$.*

PROPOSITION 13.3.9. *Let $\mathcal{M}$ be a right proper model category. If the vertical maps in the diagram*

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{X} & \longrightarrow & \widetilde{Z} & \longleftarrow & \widetilde{Y} \end{array}$$

are weak equivalences and at least one map in each row is a fibration, then the map of pullbacks $X \times_Z Y \to \widetilde{X} \times_{\widetilde{Z}} \widetilde{Y}$ is a weak equivalence.

Let us recall that the pullback of a diagram

$$\mathcal{C}_1 \xrightarrow{\mathrm{F}_1} \mathcal{D} \xleftarrow{\mathrm{F}_2} \mathcal{C}_2$$

in $\mathsf{dg}\mathcal{C}\mathsf{at}$ is given by a dg category $\mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2$ defined in the obvious way. This notion of pullback does not behave well with respect to quasi-equivalences.

Модельна структура на dg категоріях власна справа

To overcome this issue, one has to note that, by the work of Tabuada, $\mathsf{dg}\mathcal{C}\mathsf{at}$ has a model category structure whose weak equivalences are the quasi-equivalences.
Here we content ourselves with some remarks about the special case of $\mathsf{dg}\mathcal{C}\mathsf{at}$. In particular, in Tabuada's model structure all dg categories are fibrant but not all of them are cofibrant.
Furthermore, such a model structure is right proper, i.e. every pullback of a weak equivalence along a fibration is a weak equivalence, thanks to the fact that all objects are fibrant.
Finally, Hqe can be reinterpreted as the homotopy category of $\mathsf{dg}\mathcal{C}\mathsf{at}$ with respect to such a model structure.

Гомотопійне відтягування dg категорій

One can consider the homotopy pullback $\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}_2$. By definition $\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}_2 := \mathcal{C}'_1 \times_{\mathcal{D}} \mathcal{C}'_2$ is the usual pullback of a diagram

$$\mathcal{C}'_1 \xrightarrow{\mathrm{F}'_1} \mathcal{D} \xleftarrow{\mathrm{F}'_2} \mathcal{C}'_2, \tag{1}$$

where at least one among F'_1 and F'_2 is a fibration and (for $\mathrm{i} = 1, 2$) $\mathrm{F}_\mathrm{i} = \mathrm{F}'_\mathrm{i} \circ \mathrm{I}_\mathrm{i}$ with $\mathrm{I}_\mathrm{i} \colon \mathcal{C}_\mathrm{i} \to \mathcal{C}'_\mathrm{i}$ a quasi-equivalence.
Notice that such a factorization of F_i always exists, and in fact one could choose I_i to be a cofibration as well. The homotopy pullback does not depend, up to isomorphism in Hqe, on the choice of the diagram (1).
Let us spell out an explicit description of $\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}_2$. We can take $\mathrm{F}'_2 = \mathrm{F}_2$ and factor only F_1 as follows. Define $\mathcal{C}'_1$ to be the dg category whose objects are triples, $(\mathrm{C}_1, \mathrm{D}, \mathrm{f})$ where $\mathrm{C}_1 \in \mathsf{Ob}(\mathcal{C}_1)$, $\mathrm{D} \in \mathsf{Ob}(\mathcal{D})$ and $\mathrm{f} \colon \mathrm{F}_1(\mathrm{C}_1) \to \mathrm{D}$ is a strong homotopy equivalence.

Definition (Kontsevich's category again)

Objects D and D' of a dg-category $\mathcal{D}$ are strongly homotopy equivalent =strongly homotopy isomorphic if there are morphisms of $\mathcal{D}$

$$\mathrm{D} \xrightarrow[0]{\mathrm{f}} \mathrm{D}'$$

$$\alpha \underset{-1}{\circlearrowright} \mathrm{D} \xleftarrow[0]{\mathrm{g}} \mathrm{D}' \underset{-1}{\circlearrowleft} \beta$$

$$\mathrm{D} \xrightarrow[-2]{\delta} \mathrm{D}'$$

such that

$$\begin{aligned} \mathrm{g} \circ \mathrm{f} &= 1_{\mathrm{D}} - \mathrm{d}\alpha, \\ \mathrm{f} \circ \mathrm{g} &= 1_{\mathrm{D}'} - \mathrm{d}\beta, \\ \mathrm{f} \circ \alpha - \beta \circ \mathrm{f} &= \mathrm{d}\delta. \end{aligned}$$

Morphisms f and g are called strong homotopy equivalences = strong homotopy isomorphisms, homotopy inverse to each other.

Exercise

Strong homotopy equivalence is an equivalence relation. In particular, the composition of strong homotopy equivalences is a strong homotopy equivalence.

A morphism of degree n from (C_1, D, f) to (C'_1, D', f') in $\mathcal{C}'_1$ is given by a triple (a_1, b, h) with $a_1 \in \mathsf{Hom}_{\mathcal{C}_1}(C_1, C'_1)^n$, $b \in \mathsf{Hom}_{\mathcal{D}}(D, D')^n$ and $h \in \mathsf{Hom}_{\mathcal{D}}(F_1(C_1), D')^{n-1}$.
The differential is defined by

$$d(a_1, b, h) := (d(a_1), d(b), d(h) + (-1)^n(f' \circ F_1(a_1) - b \circ f))$$

and the composition by

$$(a'_1, b', h') \circ (a_1, b, h) := (a'_1 \circ a_1, b' \circ b, b' \circ h + (-1)^n h' \circ F_1(a_1)).$$

The dg functor $I_1 : \mathcal{C}_1 \to \mathcal{C}'_1$ is defined by $I_1(C_1) := (C_1, F_1(C_1), \mathsf{id}_{F_1(C_1)})$ on objects and $I_1(a_1) := (a_1, F_1(a_1), 0)$ on morphisms. On the other hand, the dg functor $F'_1 : \mathcal{C}'_1 \to \mathcal{D}$ is defined as projection on the second component both on objects and on morphisms. It is not difficult to check that I_1 is a quasi-equivalence and F'_1 is a fibration.

Гомотопійна оборотність на морфізмах

Identity morphism of $(C_1, D, f) \in \mathsf{Ob}\,\mathcal{C}_1'$ is $(1_{C_1}, 1_D, 0)$.

I_1 is a homotopy invertible on morphisms:

$$I_1 : \mathcal{C}_1(X, Y) \to \mathcal{C}_1'((X, F_1X, 1_{F_1X}), (Y, F_1Y, 1_{F_1Y}))$$
$$= \mathcal{C}_1(X, Y) \oplus \mathcal{D}(F_1X, F_1Y) \oplus \mathcal{D}(F_1X, F_1Y)[-1], \ a \mapsto (a, F_1a, 0)$$

has a homotopy inverse pr_1.
In fact, $\mathsf{pr}_1(I_1a) = a$, and $I_1(\mathsf{pr}_1(a, b, h)) = (a, F_1a, 0)$,
$(\mathrm{id} - I_1 \circ \mathsf{pr}_1)(a, b, h) = (0, b - F_1a, h)$ coincides with
$(dz + zd)(a, b, h)$, the homotopy is

$$z : \mathcal{C}_1(X, Y)^n \oplus \mathcal{D}(F_1X, F_1Y)^n \oplus \mathcal{D}(F_1X, F_1Y)^{n-1}$$
$$\to \mathcal{C}_1(X, Y)^{n-1} \oplus \mathcal{D}(F_1X, F_1Y)^{n-1} \oplus \mathcal{D}(F_1X, F_1Y)^{n-2},$$
$$(a, b, h) \mapsto (0, (-1)^n h, 0).$$

Деякі сильні гомотопійні еквівалентності

For any object $(\mathrm{C}, \mathrm{D}, \mathrm{f} : \mathrm{F}_1\mathrm{C} \to \mathrm{D})$ of $\mathcal{C}_1'$ there exists a strong homotopy equivalence $\tilde{\mathrm{f}} : (\mathrm{C}, \mathrm{F}_1\mathrm{C}, 1_{\mathrm{F}_1\mathrm{C}}) \to (\mathrm{C}, \mathrm{D}, \mathrm{f}) \in \mathcal{C}_1'$. It is given by $\tilde{\mathrm{f}} = (1_\mathrm{C}, \mathrm{f}, 0) \in \mathcal{C}_1(\mathrm{C}, \mathrm{C})^0 \oplus \mathcal{D}(\mathrm{F}_1\mathrm{C}, \mathrm{D})^0 \oplus \mathcal{D}(\mathrm{F}_1\mathrm{C}, \mathrm{D})^{-1}$, accompanied with $\tilde{\mathrm{g}} : (\mathrm{C}, \mathrm{D}, \mathrm{f}) \to (\mathrm{C}, \mathrm{F}_1\mathrm{C}, 1_{\mathrm{F}_1\mathrm{C}}) \in \mathcal{C}_1'$,
$\tilde{\mathrm{g}} = (1_\mathrm{C}, \mathrm{g}, -\alpha) \in \mathcal{C}_1(\mathrm{C}, \mathrm{C})^0 \oplus \mathcal{D}(\mathrm{D}, \mathrm{F}_1\mathrm{C})^0 \oplus \mathcal{D}(\mathrm{F}_1\mathrm{C}, \mathrm{F}_1\mathrm{C})^{-1}$,
$\tilde{\alpha} = (0, \alpha, 0) \in \mathcal{C}_1(\mathrm{C}, \mathrm{C})^{-1} \oplus \mathcal{D}(\mathrm{F}_1\mathrm{C}, \mathrm{F}_1\mathrm{C})^{-1} \oplus \mathcal{D}(\mathrm{F}_1\mathrm{C}, \mathrm{F}_1\mathrm{C})^{-2}$,
$\tilde{\beta} = (0, \beta, \delta) \in \mathcal{C}_1(\mathrm{C}, \mathrm{C})^{-1} \oplus \mathcal{D}(\mathrm{D}, \mathrm{D})^{-1} \oplus \mathcal{D}(\mathrm{F}_1\mathrm{C}, \mathrm{D})^{-2}$,
$\tilde{\delta} = (0, \delta, 0) \in \mathcal{C}_1(\mathrm{C}, \mathrm{C})^{-2} \oplus \mathcal{D}(\mathrm{F}_1\mathrm{C}, \mathrm{D})^{-2} \oplus \mathcal{D}(\mathrm{F}_1\mathrm{C}, \mathrm{D})^{-3}$,
satisfying

$$\tilde{\mathrm{g}} \circ \tilde{\mathrm{f}} = 1 - \mathrm{d}\tilde{\alpha},$$

$$\tilde{\mathrm{f}} \circ \tilde{\mathrm{g}} = 1 - \mathrm{d}\tilde{\beta},$$

$$\tilde{\mathrm{f}} \circ \tilde{\alpha} - \tilde{\beta} \circ \tilde{\mathrm{f}} = \mathrm{d}\tilde{\delta}.$$

Гомотопійні екків-сті в передтриангульованій dg-кат.

Lemma (likely known)

Let $\mathcal{D}$ be a pretriangulated dg-category. Let $f : M \to N \in Z^0\mathcal{D}$. Then f is homotopy invertible iff $\mathsf{Cone}\, f$ is contractible iff f is strongly homotopy invertible.

Доведення. Assume that $f : M \to N \in Z^0\mathcal{D}$ is homotopy invertible. The category $H^0\mathcal{D}$ is triangulated. The square

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle 1} \\ N & \xrightarrow{1} & N \end{array}$$

extends to a morphism of distinguished triangles

$$\begin{array}{ccccccc} M & \xrightarrow{f} & N & \longrightarrow & \mathsf{Cone}\, f & \longrightarrow & M[1] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f[1]} \\ N & \xrightarrow{1} & N & \longrightarrow & 0 & \longrightarrow & N[1] \end{array}$$

by property [TR3] of triangulated category $\mathrm{H}^0\mathcal{D}$.
The morphism $0 : \mathsf{Cone}\,\mathrm{f} \to 0$ is invertible in $\mathrm{H}^0\mathcal{D}$, that is,
$\mathrm{C} = \mathsf{Cone}\,\mathrm{f}$ is contractible ($1_{\mathrm{C}} = \mathrm{dh}$ for some $\mathrm{h} \in \mathcal{D}(\mathrm{C},\mathrm{C})^{-1}$).
C is accompanied by morphisms $\sigma \in \mathcal{D}(\mathrm{M},\mathrm{M}[1])^{-1}$, $\mathrm{d}\sigma = 0$,
$\sigma^{-1} \in \mathcal{D}(\mathrm{M}[1],\mathrm{M})^1$; $\mathrm{M}[1] \xrightarrow{\mathrm{i}} \mathrm{C} \xrightarrow{\mathrm{p}} \mathrm{M}[1]$, $\mathrm{N} \xrightarrow{\mathrm{j}} \mathrm{C} \xrightarrow{\mathrm{s}} \mathrm{N}$ of degree 0;
such that $\mathrm{p}\circ\mathrm{i} = 1$, $\mathrm{s}\circ\mathrm{j} = 1$, $\mathrm{s}\circ\mathrm{i} = 0$, $\mathrm{p}\circ\mathrm{j} = 0$, $\mathrm{i}\circ\mathrm{p} + \mathrm{j}\circ\mathrm{s} = 1$,
$\mathrm{dp} = 0$, $\mathrm{dj} = 0$, $\mathrm{di} = \mathrm{j}\circ\mathrm{f}\circ\sigma^{-1}$, $\mathrm{ds} = -\mathrm{f}\circ\sigma^{-1}\circ\mathrm{p}$.
Clearly, h can be recovered from morphisms
$\mathrm{g} = \sigma^{-1}\circ\mathrm{p}\circ\mathrm{h}\circ\mathrm{j} \in \mathcal{D}(\mathrm{N},\mathrm{M})^0$,
$\alpha = -\sigma^{-1}\circ\mathrm{p}\circ\mathrm{h}\circ\mathrm{i}\circ\sigma \in \mathcal{D}(\mathrm{M},\mathrm{M})^{-1}$, $\beta = \mathrm{s}\circ\mathrm{h}\circ\mathrm{j} \in \mathcal{D}(\mathrm{N},\mathrm{N})^{-1}$,
$\delta = \mathrm{s}\circ\mathrm{h}\circ\mathrm{i}\circ\sigma \in \mathcal{D}(\mathrm{M},\mathrm{N})^{-2}$ as
$\mathrm{h} = \mathrm{i}\circ\sigma\circ\mathrm{g}\circ\mathrm{s} - \mathrm{i}\circ\sigma\circ\alpha\circ\sigma^{-1}\circ\mathrm{p} + \mathrm{j}\circ\beta\circ\mathrm{s} + \mathrm{j}\circ\delta\circ\sigma^{-1}\circ\mathrm{p}$.
The equation $\mathrm{dh} = 1_{\mathrm{C}}$ can be written as the system

$$\begin{aligned}
\mathrm{dg} &= 0,\\
\mathrm{d}\alpha &= 1_{\mathrm{M}} - \mathrm{g}\circ\mathrm{f},\\
\mathrm{d}\beta &= 1_{\mathrm{N}} - \mathrm{f}\circ\mathrm{g},\\
\mathrm{d}\delta &= \mathrm{f}\circ\alpha - \beta\circ\mathrm{f}.
\end{aligned}$$

The first equation says that $g \in Z^0\mathcal{D}(N, M)$. The second and the third say that g is homotopy inverse to f.
The fourth equation says that we deal with representation of Kontsevich's category, that is, M and N are strongly homotopy equivalent. □

peut montrer aisément que pour la structure obtenue, tout objet est fibrant et qu'un dg-foncteur F de $\mathcal{C}$ vers $\mathcal{D}$ est une fibration si et seulement si :

- pour tous objects c_1 et c_2 dans $\mathcal{C}$, le morphisme de complexes de $\mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$ vers $\mathrm{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$ est surjectif en chaque composante et
- pour tout objet c_1 dans $\mathcal{C}$ et tout isomorphisme v de $F(c_1)$ vers d dans $\mathrm{H}^0(\mathcal{D})$, il existe un isomorphisme u de c_1 vers c_2 dans $\mathrm{H}^0(\mathcal{C})$ tel que $F(u) = v$.

$F_1' : (a, b, h) \mapsto b$ is surjective on morphisms.

Lemma

Let $\mathcal{D}$ be a pretriangulated **dg**-category. Then $F_1' : \mathcal{C}_1' \to \mathcal{D}$ is a fibration in Tabuada's model structure.

Доведення.

An isomorphism $v : D \to D' \in H^0\mathcal{D}$ lifts to a strong homotopy isomorphism $v : D \to D' \in \mathcal{D}$. It is lifted to a morphism $(1_C, v, 0) : (C, D, f) \to (C, D', v \circ f) \in \mathcal{C}_1'$ (notice that $v \circ f$ is a strong homotopy isomorphism) whose second projection is v. □

Явний вираз для гомотопійного відтягування dg категорій

Let $\mathcal{D}$ be a pretriangulated dg-category. With the above choice, $\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}_2$ can be identified with the dg category whose objects are triples (C_1, C_2, f), where $C_i \in \mathsf{Ob}(\mathcal{C}_i)$, for $i = 1, 2$, and $f\colon F_1(C_1) \to F_2(C_2)$ is a strong homotopy equivalence.
A morphism of degree n from (C_1, C_2, f) to (C_1', C_2', f') in $\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}_2$ is given by a triple (a_1, a_2, h) with $a_i \in \mathsf{Hom}_{\mathcal{C}_i}(C_i, C_i')^n$, for $i = 1, 2$, and $h \in \mathsf{Hom}_{\mathcal{D}}(F_1(C_1), F_2(C_2'))^{n-1}$.
The differential is defined by

$$d(a_1, a_2, h) := (d(a_1), d(a_2), d(h) + (-1)^n (f' \circ F_1(a_1) - F_2(a_2) \circ f))$$

and the composition by

$$(a_1', a_2', h') \circ (a_1, a_2, h) := (a_1' \circ a_1, a_2' \circ a_2, F_2(a_2') \circ h + (-1)^n h' \circ F_1(a_1)).$$

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