11. dg фактор-категорія Дрінфельда. Навколо похідних категорій

Володимир Любашенко

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dg категорія Дрінфельда

Let $\mathcal{A}$ be a DG category and $\mathcal{B} \subset \mathcal{A}$ a full DG subcategory. We denote by $\mathcal{A}/\mathcal{B}$ the DG category obtained from $\mathcal{A}$ by adding for every object $U \in \mathcal{B}$ a morphism $\varepsilon_U : U \to U$ of degree -1 such that $\mathrm{d}(\varepsilon_U) = \mathrm{id}_U$ (we add neither new objects nor new relations between the morphisms).

So for $X, Y \in \mathcal{A}$ we have an isomorphism of graded k-modules (but not an isomorphism of complexes)

$$\bigoplus_{n=0}^{\infty} \mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y), \tag{3.1}$$

where $\mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X, Y)$ is the direct sum of tensor products $\mathrm{Hom}_{\mathcal{A}}(U_n, U_{n+1}) \otimes k[1] \otimes \mathrm{Hom}_{\mathcal{A}}(U_{n-1}, U_n) \otimes k[1] \otimes \cdots \otimes k[1] \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(U_0, U_1)$, $U_0 := X$, $U_{n+1} := Y$, $U_i \in \mathcal{B}$ for $1 \leqslant i \leqslant n$ (in particular, $\mathrm{Hom}^0_{\mathcal{A}/\mathcal{B}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$); the morphism (3.1) maps $f_n \otimes \varepsilon \otimes f_{n-1} \otimes \cdots \otimes \varepsilon \otimes f_0$ to $f_n \varepsilon_{U_n} f_{n-1} \cdots \varepsilon_{U_1} f_0$, where ε is the canonical generator of $k[1]$. Using the formula $\mathrm{d}(\varepsilon_U) = \mathrm{id}_U$ one can easily find the differential on the l.h.s. of (3.1) corresponding to the one on the r.h.s. The image of $\bigoplus_{n=0}^{N} \mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X, Y)$ is a subcomplex of $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$, so we get a filtration on $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$. The map (3.1) induces an isomorphism of complexes

$$\bigoplus_{n=0}^{\infty} \mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X, Y) \xrightarrow{\sim} \mathrm{gr}\,\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y). \tag{3.2}$$

Узгодкеність фактор-категорій Дрінфельда та Верд'є

3.2. Example. If $\mathcal{A}$ has a single object U with $\mathrm{End}_{\mathcal{A}} U = R$ then $\mathcal{A}/\mathcal{A}$ has a single object U with $\mathrm{End}_{\mathcal{A}/\mathcal{A}} U = \widetilde{R}$, where the DG algebra $\widetilde{R}$ is obtained from the DG algebra R by adding a new generator ε of degree -1 with $d\varepsilon = 1$. As a DG R-bimodule, $\widetilde{R}$ equals $\mathrm{Cone}(\mathrm{Bar}(R) \to R)$, where $\mathrm{Bar}(R)$ is the bar resolution of the DG R-bimodule R. Both descriptions of $\widetilde{R}$ show that it has zero cohomology.

3.3. The triangulated functor $\mathcal{A}^{\mathrm{tr}} \to (\mathcal{A}/\mathcal{B})^{\mathrm{tr}}$ maps $\mathcal{B}^{\mathrm{tr}}$ to zero and therefore induces a triangulated functor $\Phi : \mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to (\mathcal{A}/\mathcal{B})^{\mathrm{tr}}$. Here $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}$ denotes Verdier's quotient (see Appendix A). We will prove that if k is a field then Φ is an equivalence. For a general ring k this is true under an additional assumption. E.g., it is enough to assume that $\mathcal{A}$ is homotopically flat over k (we prefer to use the name "homotopically flat" instead of Spaltenstein's name "K-flat" which is probably due to the notation $K(\mathcal{C})$ for the homotopy category of complexes in an additive category $\mathcal{C}$). A DG category $\mathcal{A}$ is said to be *homotopically flat* over k if for every $X, Y \in \mathcal{A}$ the complex $\mathrm{Hom}(X, Y)$ is homotopically flat over k in Spaltenstein's sense [50], i.e., for every acyclic complex C of k-modules $C \otimes_k \mathrm{Hom}(X, Y)$ is acyclic. In fact, homotopical flatness of $\mathcal{A}$ can be replaced by one of the following weaker assumptions:

$$\mathrm{Hom}(X, U) \text{ is homotopically flat over } k \text{ for all } X \in \mathcal{A},\ U \in \mathcal{B}; \tag{3.3}$$

$$\mathrm{Hom}(U, X) \text{ is homotopically flat over } k \text{ for all } X \in \mathcal{A},\ U \in \mathcal{B}. \tag{3.4}$$

3.4. Theorem. *Let $\mathcal{A}$ be a DG category and $\mathcal{B} \subset \mathcal{A}$ a full DG subcategory. If either* (3.3) *or* (3.4) *holds then $\Phi : \mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to (\mathcal{A}/\mathcal{B})^{\mathrm{tr}}$ is an equivalence.*

3.5. If (3.3) and (3.4) are not satisfied one can construct a diagram (1.1) by choosing a homotopically flat resolution $\widetilde{\mathcal{A}} \xrightarrow{\approx} \mathcal{A}$ and putting $\mathcal{C} := \widetilde{\mathcal{A}}/\widetilde{\mathcal{B}}$, where $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{A}}$ is the full subcategory of objects whose image in $\mathcal{A}$ is homotopy equivalent to an object of $\mathcal{B}$. Here "homotopically flat resolution" means that $\widetilde{\mathcal{A}}$ is homotopically flat and the DG functor $\widetilde{\mathcal{A}} \to \mathcal{A}$ is a quasi-equivalence (see 2.3). The existence of homotopically flat resolutions of $\mathcal{A}$ follows from Lemma B.5.

(ii) Let $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ be as in 3.5 and suppose that (3.3) or (3.4) holds for both $\mathcal{B} \subset \mathcal{A}$ and $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{A}}$. Then the DG functor $\widetilde{\mathcal{A}}/\widetilde{\mathcal{B}} \to \mathcal{A}/\mathcal{B}$ is a quasi-equivalence, i.e., it induces an equivalence of the corresponding homotopy categories. This follows from Theorem 3.4. One can also directly show that if $X, Y \in \mathrm{Ob}(\mathcal{A}/\mathcal{B}) = \mathrm{Ob}\,\mathcal{A}$ are the images of $\widetilde{X}, \widetilde{Y} \in \mathrm{Ob}(\widetilde{\mathcal{A}}/\widetilde{\mathcal{B}}) = \mathrm{Ob}\,\widetilde{\mathcal{A}}$ then the morphism $\mathrm{Hom}_{\widetilde{\mathcal{A}}/\widetilde{\mathcal{B}}}(\widetilde{X}, \widetilde{Y}) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$ is a quasi-isomorphism (use (3.2) and notice that the morphism $\mathrm{Hom}^n_{\widetilde{\mathcal{A}}/\widetilde{\mathcal{B}}}(\widetilde{X}, \widetilde{Y}) \to \mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X, Y)$ is a quasi-isomorphism for every n; this follows directly from the definition of Hom^n and the fact that (3.3) or (3.4) holds for $\mathcal{B} \subset \mathcal{A}$ and $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{A}}$).

Let $A \in \mathsf{dg}$. The map $\sigma : A \to A[1]$ graded commutes with the differential:

$d \otimes 1 + 1 \otimes d$

$$\begin{array}{ccc} A & \xrightarrow{\quad\sigma\quad} & A[1] \\ {\scriptstyle d}\downarrow & - & \downarrow{\scriptstyle d} \\ A & \xrightarrow{\quad\sigma\quad} & A[1] \end{array}$$

Let $A, B \in \mathsf{dg}$. We have isomorphisms of complexes

$$\begin{array}{ccccc} (A \otimes B)[1] & \xrightarrow{\sigma^{-1}} & A \otimes B & \xrightarrow{1 \otimes \sigma} & A \otimes (B[1]) \\ {\scriptstyle d}\downarrow & - & \downarrow{\scriptstyle d} & - & \downarrow{\scriptstyle d} \\ (A \otimes B)[1] & \xrightarrow{\sigma^{-1}} & A \otimes B & \xrightarrow{1 \otimes \sigma} & A \otimes (B[1]) \end{array}$$

$$\begin{array}{ccccc} (A \otimes B)[1] & \xrightarrow{\sigma^{-1}} & A \otimes B & \xrightarrow{\sigma \otimes 1} & A[1] \otimes B \\ {\scriptstyle d}\downarrow & - & \downarrow{\scriptstyle d} & - & \downarrow{\scriptstyle d} \\ (A \otimes B)[1] & \xrightarrow{\sigma^{-1}} & A \otimes B & \xrightarrow{\sigma \otimes 1} & A[1] \otimes B \end{array}$$

On components $\sigma^{-1} \cdot (1 \otimes \sigma)$ is the identity map:

$$\{(A \otimes B)[1]\}^n = \oplus_{k+p=n+1} A^k \otimes B^p$$
$$= \oplus_{k+p-1=n} A^k \otimes B^p = \oplus_{k+m=n} A^k \otimes B^{1+m} = \{A \otimes (B[1])\}^n.$$

Lemma

$B \to C$ $\quad$ A

Let $p : B \to C \in \mathsf{dg}$, $A \in \mathsf{dg}$. Then there is an isomorphism of complexes $\mathsf{Cone}(1_A \otimes p) \cong A \otimes \mathsf{Cone}\, p$.

$: A \otimes B \to A \otimes C$

Доведення.

$$\mathsf{Cone}(1_A \otimes p) = \left((A \otimes B)[1] \oplus A \otimes C, \begin{pmatrix} d_{(A\otimes B)[1]} & \sigma^{-1}(1_A \otimes p) \\ 0 & d_{A\otimes C} \end{pmatrix} \right)$$

$$\Big\downarrow \sigma^{-1}\cdot(1\otimes\sigma)\oplus 1 \quad \cong$$

$$\left(A \otimes (B[1]) \oplus A \otimes C, \begin{pmatrix} d_A \otimes 1_{B[1]} + 1_A \otimes d_{B[1]} & 1_A \otimes \sigma^{-1} p \\ 0 & d_A \otimes 1_C + 1_A \otimes d_C \end{pmatrix} \right)$$

$$\Big\downarrow \cong$$

$$A \otimes \mathsf{Cone}\, p = \left(A \otimes (B[1] \oplus C), d_A \otimes \begin{pmatrix} 1_{B[1]} & 0 \\ 0 & 1_C \end{pmatrix} + 1_A \otimes \begin{pmatrix} d_{B[1]} & \sigma^{-1} p \\ 0 & d_C \end{pmatrix} \right)$$

Lemma

Let $q : A \to C \in \mathsf{dg}$, $B \in \mathsf{dg}$. Then there is an isomorphism of complexes $\mathsf{Cone}(q \otimes 1_B) \cong (\mathsf{Cone}\, q) \otimes B$.

Доведення.

$$\mathsf{Cone}(q \otimes 1_B) = \left((A \otimes B)[1] \oplus C \otimes B, \begin{pmatrix} d_{(A\otimes B)[1]} & \sigma^{-1}(q \otimes 1_B) \\ 0 & d_{C\otimes B} \end{pmatrix} \right)$$

$$\Big\downarrow \sigma^{-1}\cdot(\sigma\otimes 1)\oplus 1 \;\cong$$

$$\left(A[1] \otimes B \oplus C \otimes B, \begin{pmatrix} d_{A[1]} \otimes 1_B + 1_{A[1]} \otimes d_B & (\sigma^{-1} q) \otimes 1_B \\ 0 & d_C \otimes 1_B + 1_C \otimes d_B \end{pmatrix} \right)$$

$$\Big\downarrow \cong$$

$$(\mathsf{Cone}\, q) \otimes B = \left((A[1] \oplus C) \otimes B, \begin{pmatrix} d_{A[1]} & \sigma^{-1} q \\ 0 & d_C \end{pmatrix} \otimes 1_B + \begin{pmatrix} 1_{A[1]} & 0 \\ 0 & 1_C \end{pmatrix} \otimes d_B \right)$$

□

Corollary

Let $p : B \to C \in \mathsf{dg}$, $A, D \in \mathsf{dg}$. Then there is an isomorphism of complexes $\mathsf{Cone}(1_A \otimes p \otimes 1_D) \cong A \otimes (\mathsf{Cone}\, p) \otimes D$.

Corollary

Let $p : B \to C \in \mathsf{dg}$ be a quasi-isomorphism and let $A, D \in \mathsf{dg}$ be homotopy flat. Then $1_A \otimes p \otimes 1_D$ is a quasi-isomorphism.

Proposition

Let $\mathcal{A}$ be a locally homotopy flat dg-category. Then $\mathcal{C} = \mathcal{A}\langle f_i, df_i \in \mathcal{A}^\bullet \mid i \in I\rangle$ is locally homotopy flat.

Доведення.

$\mathsf{Ob}\,\mathcal{C} = \mathsf{Ob}\,\mathcal{A}$, $\mathcal{C}(X,Y) = \oplus_{n=0}^{\infty} \mathsf{Hom}^n_{\mathcal{C}}(X,Y)$, where $\mathsf{Hom}^0_{\mathcal{C}} = \mathcal{A}$,

$\Bbbk[-\deg f_i]$

$$\mathsf{Hom}^n_{\mathcal{C}}(X,Y) = \oplus_{i_1,\dots,i_n \in I} \mathcal{A}(X, \mathsf{src}\, f_{i_1}) \otimes \Bbbk f_{i_1} \otimes \mathcal{A}(\mathsf{tgt}\, f_{i_1}, \mathsf{src}\, f_{i_2}) \otimes \Bbbk f_{i_2} \otimes \mathcal{A}(\mathsf{tgt}\, f_{i_2}, \mathsf{src}\, f_{i_3}) \otimes \dots \otimes \Bbbk f_{i_n} \otimes \mathcal{A}(\mathsf{tgt}\, f_{i_n}, Y).$$

$\mathcal{A}$ – locally homotopy flat $\Rightarrow$ complex $\mathsf{Hom}^n_{\mathcal{C}}(X,Y)$ is htpy flat.

$$0 \to \oplus_{n=0}^{N-1} \mathsf{Hom}^n_{\mathcal{C}}(X,Y) \to \oplus_{n=0}^{N} \mathsf{Hom}^n_{\mathcal{C}}(X,Y) \to \mathsf{Hom}^N_{\mathcal{C}}(X,Y) \to 0$$

is a semi-split exact sequence. $\Rightarrow$ For any acyclic $C \in \mathsf{dg}$

$$0 \to C \otimes \bigoplus_{n=0}^{N-1} \mathsf{Hom}^n_{\mathcal{C}}(X,Y) \to C \otimes \bigoplus_{n=0}^{N} \mathsf{Hom}^n_{\mathcal{C}}(X,Y) \to C \otimes \mathsf{Hom}^N_{\mathcal{C}}(X,Y) \to 0$$

is a semi-split exact sequence.

$\Rightarrow$ By induction $\oplus_{n=0}^{N} \mathsf{Hom}^n_{\mathcal{C}}(X,Y)$ is homotopy flat.
$\Rightarrow$ $\mathcal{C}(X,Y) = \oplus_{n=0}^{\infty} \mathsf{Hom}^n_{\mathcal{C}}(X,Y)$ is homotopy flat. □

Corollary

Any semi-free dg-category $\tilde{\mathcal{A}}$ is locally homotopy flat.

$\forall$ small $\mathcal{A} \in \mathsf{dg}\mathcal{C}\mathsf{at}$ $\exists$ semi-free $\tilde{\mathcal{A}}$ with $\mathsf{Ob}\,\tilde{\mathcal{A}} = \mathsf{Ob}\,\mathcal{A}$,
$\exists$ dg-functor $p : \tilde{\mathcal{A}} \to \mathcal{A}$ with $\mathsf{Ob}\,p = \mathsf{id}_{\mathsf{Ob}\,\mathcal{A}}$,
p – surjective quasi-isomorphism on morphisms.
Having full $\mathcal{B} \subset \mathcal{A}$ define $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$ by $\mathsf{Ob}\,\tilde{\mathcal{B}} = \mathsf{Ob}\,\mathcal{B}$. Corollary 4 implies that
$p \otimes 1 \otimes p \otimes 1 \otimes \cdots \otimes p : \mathsf{Hom}^n_{\tilde{\mathcal{A}}/\tilde{\mathcal{B}}}(X,Y) \to \mathsf{Hom}^n_{\mathcal{A}/\mathcal{B}}(X,Y)$ is a quasi-isomorphism. From

$$\begin{array}{ccccccccc}
0 \longrightarrow & \oplus_{n=0}^{N-1} \mathsf{Hom}^n_{\tilde{\mathcal{A}}/\tilde{\mathcal{B}}}(X,Y) & \longrightarrow & \oplus_{n=0}^{N} \mathsf{Hom}^n_{\tilde{\mathcal{A}}/\tilde{\mathcal{B}}}(X,Y) & \longrightarrow & \mathsf{Hom}^N_{\tilde{\mathcal{A}}/\tilde{\mathcal{B}}}(X,Y) & \longrightarrow 0 \\
& \text{qis}\downarrow & & \downarrow & & \downarrow\text{qis} & \\
0 \longrightarrow & \oplus_{n=0}^{N-1} \mathsf{Hom}^n_{\mathcal{A}/\mathcal{B}}(X,Y) & \longrightarrow & \oplus_{n=0}^{N} \mathsf{Hom}^n_{\mathcal{A}/\mathcal{B}}(X,Y) & \longrightarrow & \mathsf{Hom}^N_{\mathcal{A}/\mathcal{B}}(X,Y) & \longrightarrow 0
\end{array}$$

we deduce that the middle vertical map is a quasi-isomorphism.

3.7.1. Let $\mathcal{A}_0$ be the DG category with two objects X_1, X_2 freely generated by a morphism $f : X_1 \to X_2$ of degree 0 with $\mathrm{d}f = 0$ (so $\mathrm{Hom}(X_i, X_i) = k$, $\mathrm{Hom}(X_1, X_2)$ is the free module kf and $\mathrm{Hom}(X_2, X_1) = 0$). Put $\mathcal{A} := \mathcal{A}_0^{\text{pre-tr}}$. Let $\mathcal{B} \subset \mathcal{A}$ be the full DG subcategory with a single object $\mathrm{Cone}(f)$. Instead of describing the whole DG quotient $\mathcal{A}/\mathcal{B}$, we will describe only the full DG subcategory $(\mathcal{A}/\mathcal{B})_0 \subset \mathcal{A}/\mathcal{B}$ with objects X_1 and X_2 (the DG functor $(\mathcal{A}/\mathcal{B})_0^{\text{pre-tr}} \to (\mathcal{A}/\mathcal{B})^{\text{pre-tr}}$ is a DG equivalence in the sense of 2.3, so $\mathcal{A}/\mathcal{B}$ can be considered as a full DG subcategory of $(\mathcal{A}/\mathcal{B})_0^{\text{pre-tr}}$). Directly using the definition of $\mathcal{A}/\mathcal{B}$ (see 3.1), one shows that $(\mathcal{A}/\mathcal{B})_0$ equals the DG category $\mathcal{K}$ freely generated by our original $f : X_1 \to X_2$ and also a morphism $g : X_2 \to X_1$ of degree 0, morphisms $\alpha_i : X_i \to X_i$ of degree -1, and a morphism $u : X_1 \to X_2$ of degree -2 with the differential given by $\mathrm{d}f = \mathrm{d}g = 0$, $\mathrm{d}\alpha_1 = gf - 1$, $\mathrm{d}\alpha_2 = fg - 1$, $\mathrm{d}u = f\alpha_1 - \alpha_2 f$. On the other hand, one has the following description of $\mathrm{Ho}^{\cdot}((\mathcal{A}/\mathcal{B})_0)$.

Vladimir G. Drinfeld, DG quotients of DG categories, J. Algebra 272 (2004), no. 2, 643–691, arXiv:math.KT/0210114 §3.1 – 3.5, 3.6(ii), 3.7.1