10. Ідемпотентне поповнення триангульованої категорії.
Навколо похідних категорій

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1.1. DEFINITION. An additive category K is said to be *idempotent complete* if any idempotent $e: A \to A$, $e^2 = e$, arises from a splitting of A,

$$A = \mathrm{Im}(e) \oplus \mathrm{Ker}(e).$$

1.2. DEFINITION. Let K be an additive category. The *idempotent completion* of K is the category $\tilde{K}$ defined as follows. Objects of $\tilde{K}$ are pairs (A, e) where A is an object of K and $e: A \to A$ is an idempotent. A morphism in $\tilde{K}$ from (A, e) to (B, f) is a morphism $\alpha: A \to B$ in K such that

$$\alpha e = f\alpha = \alpha.$$

The assignment $A \mapsto (A, 1)$ defines a functor ι from K to $\tilde{K}$. The following result is well-known.

1.3. PROPOSITION. *The category $\tilde{K}$ is additive, the functor $\iota: K \to \tilde{K}$ is additive, and $\tilde{K}$ is idempotent complete. Moreover, the functor ι induces an equivalence*

$$\mathrm{Hom}_{\mathrm{add}}(\tilde{K}, L) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{add}}(K, L)$$

for each idempotent complete additive category L, where $\mathrm{Hom}_{\mathrm{add}}$ denotes the (large) category of additive functors.

of K as a full subcategory of $\tilde{K}$. We will write "$A \in K$" to mean that A is isomorphic to an object of K.

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1.5. THEOREM. *Let K be a triangulated category. Then its idempotent completion $\tilde{K}$ admits a unique structure of triangulated category such that the canonical functor $\iota\colon K \to \tilde{K}$ becomes exact. If $\tilde{K}$ is endowed with this structure, then for each idempotent complete triangulated category L the functor ι induces an equivalence*

$$\mathrm{Hom}_{\mathrm{exact}}(\tilde{K}, L) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{exact}}(K, L),$$

where $\mathrm{Hom}_{\mathrm{exact}}$ *denotes the (large) category of exact functors.*

1.10. DEFINITION. Let K be a triangulated category. Let us denote by $T\colon K \to K$ its translation functor. Define $T\colon \tilde{K} \to \tilde{K}$ by $T(A, e) = (T(A), T(e))$. Clearly, $T \circ \iota = \iota \circ T$.

Define a triangle in $\tilde{K}$,

$$(\Delta) \qquad A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A),$$

to be *exact* when it is a direct factor of an exact triangle of K, that is, when there is an exact triangle Δ' of K and triangle maps $s\colon \Delta \to \Delta'$ and $r\colon \Delta' \to \Delta$ with $r \circ s = 1_{\Delta}$, or, equivalently, when there is a triangle Δ'' in $\tilde{K}$ such that $\Delta \oplus \Delta''$ is isomorphic to an exact triangle in K.

1.13. LEMMA. *Let be given a commutative diagram in a pre-triangulated category L,*

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & & & \downarrow{\scriptstyle T(p)} \\ A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & T(A), \end{array}$$

in which the rows are exact triangles. Suppose $p = p^2$ and $q = q^2$ are idempotents. Then there is an idempotent $r = r^2\colon C \to C$ such that the diagram

$$(*) \qquad \begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle T(p)} \\ A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & T(A) \end{array}$$

commutes.

1.14. *Proof.* By (TR3) there is a $c\colon C \to C$ making $(*)$ commutative with c instead of r. Of course, c^2 also makes the above diagram commute and so the difference $h := c^2 - c$ has the trivial square

$$h^2 = 0.$$

This is quite classical but let us remind the reader of the proof. From $hv = (c^2 - c)v = 0$, we can factor h through w; i.e., there exists $\overline{h}\colon T(A) \to C$ such that $h = \overline{h}w$ and then $h^2 = \overline{h}wh = 0$ since $wh = w(c^2 - c) = 0$.

Applying the trick of lifting idempotents, we set

$$r = c + h - 2ch.$$

Observe that c and h commute. From $h^2 = 0$, we get $r^2 = c^2 + 2ch - 4c^2h$ and then by replacing c^2 by $c + h$ we have $r^2 = c + h + 2ch - 4ch = r$, using again $h^2 = 0$. Clearly, r can replace c in the above diagram, since $hv = 0$ and $wh = 0$. By our computation, r is an idempotent. ▮

$$\begin{array}{ccccc} \mathrm{L(TA, C)} & \xrightarrow{\mathrm{L(w,C)}} & \mathrm{L(C, C)} & \xrightarrow{\mathrm{L(v,C)}} & \mathrm{L(B, C)} \\ \bar{\mathrm{h}} & \longmapsto & \bar{\mathrm{h}} \circ \mathrm{w} = \mathrm{h} & \longmapsto & \mathrm{h} \circ \mathrm{v} = 0 \end{array}$$

$$\mathrm{r}^2 = \mathrm{c} + \mathrm{h} + 2\mathrm{ch} - 4(\mathrm{c} + \mathrm{h})\mathrm{h} = \mathrm{c} + \mathrm{h} - 2\mathrm{ch} = \mathrm{r}.$$

Définition 1.1.1. Une *catégorie triangulée* $\mathcal{D}$ est une $\mathbf{Z}$-catégorie additive stricte (chap. I, 1.5.11) munie d'un ensemble de triangles (chap. I, 3.2.1), appelés *triangles distingués*, possédant les propriétés suivantes :

TRI : Tout triangle de $\mathcal{D}$ isomorphe à un triangle distingué est un triangle distingué. Pour tout objet X de $\mathcal{D}$, le triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ est distingué. Tout morphisme $u : X \to Y$ de $\mathcal{D}$ est contenu dans un triangle distingué $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$.

TRII : Un triangle de $\mathcal{D}$: $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ est distingué si et seulement si le triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ est distingué.

TRIII : Pour tout couple de triangles distingués :

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \quad ,$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$$

et tout diagramme commutatif :

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{u'} & Y' \end{array} \quad ,$$

il existe un morphisme $h : Z \to Z'$ tel que (f, g, h) soit un morphisme de triangle, *i.e.* tel que le diagramme ci-après soit commutatif :

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array} \quad .$$

(TR1) Any triangle isomorphic to an exact triangle is exact. This follows directly from the definition. If A is an object of $\tilde{K}$, there exists A' such that $A \oplus A' \in K$ (namely, if $A = (B, e)$ take $A' = (B, 1 - e)$ and check that $A \oplus A' \cong \iota(B)$). Then exactness of

$$A \oplus A' \xrightarrow{1} A \oplus A' \longrightarrow 0 \longrightarrow T(A) \oplus T(A')$$

in K insures the exactness of $A \xrightarrow{1} A \longrightarrow 0 \longrightarrow T(A)$ in $\tilde{K}$, by definition. We still have to check that any morphism fits into an exact triangle.

Let $\alpha\colon A \to B$ be a morphism in $\tilde{K}$. Let A' and B' be such that $A \oplus A' \in K$ and $B \oplus B' \in K$. Let

$$A \oplus A' \xrightarrow{a} B \oplus B' \xrightarrow{a_1} D \xrightarrow{a_2} T(A \oplus A') \qquad (1)$$

be an exact triangle in K, where $a = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Now, using Lemma 1.13 in K, we complete the following left commutative square into a morphism of exact triangles in K such that $p = p^2\colon D \to D$ is an idempotent:

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{a} & B \oplus B' & \xrightarrow{a_1} & D & \xrightarrow{a_2} & T(A \oplus A') \\
\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\big\downarrow & & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\big\downarrow & & p\big\downarrow & & \big\downarrow\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\
A \oplus A' & \xrightarrow[a]{} & B \oplus B' & \xrightarrow[a_1]{} & D & \xrightarrow[a_2]{} & T(A \oplus A').
\end{array}$$

Then $A \xrightarrow{a} B \xrightarrow{pa_1} \mathrm{Im}(p) \xrightarrow{a_2 p} T(A)$ is an exact triangle as it is a direct factor of (1).

(TR2) This is direct from the definition.

(TR3) Consider a partial map $(\alpha, \beta)\colon \Delta \to \Gamma$ of exact triangles in $\tilde{K}$ (the left square commutes):

$$\begin{array}{ccccccccc} \Delta & & A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & T(A) \\ {\scriptstyle (\alpha,\beta)}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & & & \downarrow{\scriptstyle T(\alpha)} \\ \Gamma & & X & \xrightarrow[x]{} & Y & \xrightarrow[y]{} & Z & \xrightarrow[z]{} & T(X). \end{array}$$

By definition, there are maps of triangles $i\colon \Delta \to \Delta'$, $p\colon \Delta' \to \Delta$, $j\colon \Gamma \to \Gamma'$, $q\colon \Gamma' \to \Gamma$ such that $pi = 1_\Delta$, $qj = 1_\Gamma$, and such that Δ' and Γ' are exact triangles in K. The partial map of triangles (α, β) induces a partial map of triangles $j \circ (\alpha, \beta) \circ p$ from Δ' to Γ'. Since the two latter triangles are exact in K we can apply (TR3) to extend the partial map of triangles $j \circ (\alpha, \beta) \circ p$ to a real map $a\colon \Delta' \to \Gamma'$ of triangles. Then $q \circ a \circ i\colon \Delta \to \Gamma$ is a map of triangles extending (α, β).

So far, we have established that $\tilde{K}$ is a pretriangulated category, in the sense that it satisfies all the axioms but the octahedron axiom (TR4).

TRIV : Pour tout diagramme commutatif :

$$\begin{array}{ccccc} & & X_2 & & \\ & \overset{u_3}{\nearrow} & & \overset{u_1}{\searrow} & \\ X_1 & & \xrightarrow{\quad u_2 \quad} & & X_3 \end{array}$$

et tout triplet de triangles distingués :

$$X_1 \xrightarrow{u_3} X_2 \xrightarrow{v_3} Z_3 \xrightarrow{w_3} X_1[1] \quad ,$$

$$X_2 \xrightarrow{u_1} X_3 \xrightarrow{v_1} Z_1 \xrightarrow{w_1} X_2[1] \quad ,$$

$$X_1 \xrightarrow{u_2} X_3 \xrightarrow{v_2} Z_2 \xrightarrow{w_2} X_1[1] \quad ,$$

il existe deux morphismes :

$$m_1 : Z_3 \longrightarrow Z_2 \ ,$$

$$m_3 : Z_2 \longrightarrow Z_1 \ ,$$

tels que $(\mathsf{id}_{X_1}, u_1, m_1)$ et $(u_3, \mathsf{id}_{X_3}, m_3)$ soient des morphismes de triangles, et tels que le triangle :

$$Z_3 \xrightarrow{m_1} Z_2 \xrightarrow{m_3} Z_1 \xrightarrow{v_3[1]w_1} Z_3[1]$$

soit distingué.

(TR4) *Octahedron*. Let $u: X \to Y$ and $v: Y \to Z$ be two composable morphisms. Let $w = v \circ u$ and choose exact triangles on u, v, and w in $\tilde{K}$:

$$X \xrightarrow{u} Y \xrightarrow{u_1} U \xrightarrow{u_2} T(X) \tag{1}$$

$$Y \xrightarrow{v} Z \xrightarrow{v_1} V \xrightarrow{v_2} T(Y) \tag{2}$$

$$X \xrightarrow{w} Z \xrightarrow{w_1} W \xrightarrow{w_2} T(X). \tag{3}$$

Choose A, B, and C in $\tilde{K}$ such that $X \oplus A \in K$, $Y \oplus B \in K$, and $Z \oplus C \in K$. Add to (1) the trivial triangles $A \longrightarrow 0 \longrightarrow T(A) \xrightarrow{1} T(A)$ and $0 \longrightarrow B \xrightarrow{1} B \longrightarrow 0$ to obtain the following triangle which is exact in $\tilde{K}$ (cf. Lemma 1.6):

$$X \oplus A \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}} Y \oplus B \xrightarrow{\begin{pmatrix} u_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} U \oplus B \oplus T(A)$$
$$\xrightarrow{\begin{pmatrix} u_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} T(X) \oplus T(A). \tag{4}$$

Observe that the first morphism of (4) is in K and therefore fits into an exact triangle of K which is, via ι, an exact triangle of $\tilde{K}$. Those two triangles are isomorphic since $\tilde{K}$ is pretriangulated. Therefore, (4) is isomorphic to an exact triangle of K.

Similarly, the two following triangles are isomorphic to exact triangles of K.

$$Y \oplus B \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}} Z \oplus C \xrightarrow{\begin{pmatrix} v_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} V \oplus C \oplus T(B) \xrightarrow{\begin{pmatrix} v_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} T(Y) \oplus T(B) \tag{5}$$

$$X \oplus A \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}} Z \oplus C \xrightarrow{\begin{pmatrix} w_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} W \oplus C \oplus T(A) \xrightarrow{\begin{pmatrix} w_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} T(X) \oplus T(A) \tag{6}$$

Let us put them into an octahedron (cf. Fig. 1). The octahedron exists because it is isomorphic to an octahedron in K. In particular, we find $f: U \oplus B \oplus T(A) \to W \oplus C \oplus T(A)$ and $g: W \oplus C \oplus T(A) \to V \oplus C \oplus T(B)$ which fit into the diagram of Fig. 1. The 0's and 1's appearing in f and g come from the commutativities required by the octahedron axiom.

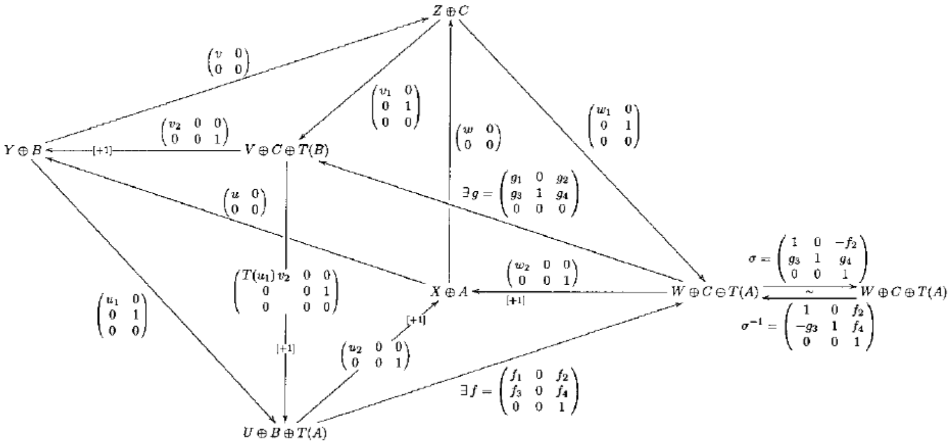


FIGURE 1

Moreover, we have

$$g_1 w_1 = v_1 \tag{7}$$

$$w_2 f_1 = u_2 \tag{8}$$

$$f_1 u_1 = w_1 v \tag{9}$$

$$v_2 g_1 = T(u) w_2. \tag{10}$$

From the relation $gf = 0$ we obtain

$$g_1 f_2 + g_2 = 0 \tag{11}$$

$$g_3 f_1 + f_3 = 0 \tag{12}$$

$$g_3 f_2 + f_4 + g_4 = 0. \tag{13}$$

We shall now use the endomorphism of $W \oplus C \oplus T(A)$,

$$\sigma := \begin{pmatrix} 1 & 0 & -f_2 \\ g_3 & 1 & g_4 \\ 0 & 0 & 1 \end{pmatrix},$$

as presented in Fig. 1, in order to modify our octahedron. Direct computation gives

$$\begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & g_3 f_2 + g_4 + f_4 \\ 0 & 0 & 1 \end{pmatrix} \overset{(13)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and similarly

$$\sigma \cdot \begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & g_3 f_2 + f_4 + g_4 \\ 0 & 0 & 1 \end{pmatrix} \overset{(13)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which implies that σ is an automorphism with the inverse

$$\sigma^{-1} = \begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

Let us modify up to isomorphism the foreground triangle of Fig. 1 by using this automorphism σ to obtain the candidate triangle

$$U \oplus B \oplus T(A) \xrightarrow{\sigma f} W \oplus C \oplus T(A) \xrightarrow{g\sigma^{-1}} V \oplus C \oplus T(B)$$
$$\xrightarrow{\begin{pmatrix} T(u_1)v_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} T(U) \oplus T(B) \oplus T^2(A), \qquad (15)$$

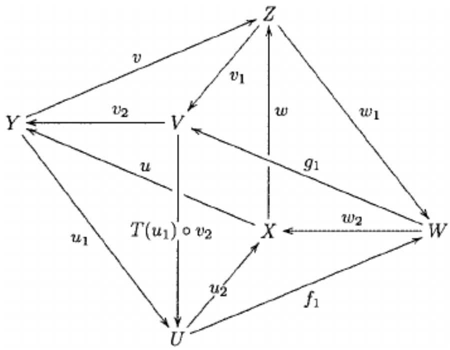
which by its construction is isomorphic to an exact triangle of K. We compute directly

$$\sigma f = \begin{pmatrix} 1 & 0 & -f_2 \\ g_3 & 1 & g_4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_1 & 0 & f_2 \\ f_3 & 0 & f_4 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} f_1 & 0 & 0 \\ g_3 f_1 + f_3 & 0 & g_3 f_2 + f_4 + g_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by (12) and (13). Similarly, we have

$$g\sigma^{-1} \overset{(14)}{=} \begin{pmatrix} g_1 & 0 & g_2 \\ g_3 & 1 & g_4 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} g_1 & 0 & g_1 f_2 + g_2 \\ 0 & 1 & g_3 f_2 + f_4 + g_4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

by (11) and (13). Putting all this together, we obtain the picture in $\tilde{K}$



in which all commutativities to be an octahedron are satisfied (use relations (7)–(10)). The only point is to check that the triangle

$$U \xrightarrow{f_1} W \xrightarrow{g_1} V \xrightarrow{T(u_1)v_2} T(U)$$

is exact in $\tilde{K}$. But this is immediate from the exact triangle (15), the explicit computations of σf, and $g\sigma^{-1}$, and from Definition 1.10. ∎

1.17. *Proof of Theorem* 1.5. Clearly, by construction of the triangulation on $\tilde{K}$, the functor $\iota\colon K \to \tilde{K}$ is exact (see Remark 1.11). By Lemma 1.6, any triangulation on $\tilde{K}$ has to contain the class of exact triangles given in Definition 1.10. It is a well-known fact that there cannot exist two different triangulated structures on an additive category such that one of them contains the other (easy consequence of TR1–TR3). This proves the uniqueness of the triangulated structure. The rest follows from Proposition 1.3 once we have shown that any additive functor $f\colon \tilde{K} \to L$ is exact as soon as $f \circ \iota$ is exact. But this is an immediate consequence of Lemma 1.6 and Definition 1.10. ∎

- Paul Balmer and Marco Schlichting, Idempotent completion of triangulated categories, J. Algebra 236 (2001), no. 2, 819–834. §1
- Jean-Louis Verdier, Des catégories dérivées des catégories abéliennes, Astérisque (1996), no. 239, xii+253 pp., With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. Chap. II, Définition 1.1.1.