8. Гомотопічно проективні модулі. Навколо похідних категорій

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Конус квазі-ізоморфізма – ациклічний

Let $\mathcal{C}$ be a dg-category over a commutative ring $\Bbbk$.

Lemma (well-known)

Let $g : A \to B \in Z^0 \mathrm{dgMod}\text{-}\mathcal{C}$. Then g is a quasi-isomorphism iff $\mathsf{Cone}\, g$ is acyclic.

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Доведення.

For all $X \in \mathsf{Ob}\,\mathcal{C}$ the functor $A \mapsto H^0(A(X))$ is cohomological. The distinguished triangle $A \xrightarrow{g} B \to \mathsf{Cone}\, g \to$ in $H^0\mathrm{dgMod}\text{-}\mathcal{C}$ implies exact sequence

$$\begin{aligned} H^{k-1}(\mathsf{Cone}\, g(X)) \to H^k(A(X)) \to H^k(B(X)) \\ \to H^k(\mathsf{Cone}\, g(X)) \to H^{k+1}(A(X)) \to H^{k+1}(B(X)). \end{aligned}$$

Vanishing of $H^\bullet(\mathsf{Cone}\, g(X))$ is equivalent to $H^\bullet(g(X))$ being an isomorphism. $\square$

Конус гомотопійно оборотного морфізма – стягуваний

Lemma (well-known)

Let $f : M \to N \in Z^0\text{dgMod-}\mathcal{C}$. Then f is homotopy invertible iff $\mathsf{Cone}\, f$ is contractible.

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Let $f : M \to N \in Z^0\mathrm{dgMod}\text{-}\mathcal{C}$. Then f is homotopy invertible iff $\mathsf{Cone}\, f$ is contractible.

Доведення. Assume that $\mathsf{Cone}\, f = (M[1] \oplus N, d_{\mathsf{Cone}\, f})$ is contractible,

$$d_{\mathsf{Cone}\, f} = \begin{pmatrix} d_M[1] & \sigma^{-1} \cdot f \\ 0 & d_N \end{pmatrix}.$$

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The contracting homotopy $h \in \mathrm{dgMod}\text{-}\mathcal{C}(\mathsf{Cone}\, f, \mathsf{Cone}\, f)^{-1}$ can be presented as

$$h = \begin{pmatrix} \alpha[1] & \sigma^{-1} \cdot \delta \\ g \cdot \sigma & \beta \end{pmatrix},$$

where $g \in \mathrm{dgMod}\text{-}\mathcal{C}(N, M)^0$,$\alpha \in \mathrm{dgMod}\text{-}\mathcal{C}(M, M)^{-1}$, $\beta \in \mathrm{dgMod}\text{-}\mathcal{C}(N, N)^{-1}$, $\delta \in \mathrm{dgMod}\text{-}\mathcal{C}(M, N)^{-2}$.

The equation $\mathrm{d}_{\mathsf{Cone}\,\mathrm{f}}\mathrm{h} + \mathrm{h}\mathrm{d}_{\mathsf{Cone}\,\mathrm{f}} = 1_{\mathsf{Cone}\,\mathrm{f}}$ can be written as the system

$$\begin{aligned}
\mathrm{d}_{\mathrm{N}}\mathrm{g} &= \mathrm{g}\mathrm{d}_{\mathrm{M}},\\
1_{\mathrm{M}} - \mathrm{f}\cdot\mathrm{g} &= \mathrm{d}_{\mathrm{M}}\alpha + \alpha\mathrm{d}_{\mathrm{M}},\\
1_{\mathrm{N}} - \mathrm{g}\cdot\mathrm{f} &= \mathrm{d}_{\mathrm{N}}\beta + \beta\mathrm{d}_{\mathrm{N}},\\
\alpha\cdot\mathrm{f} - \mathrm{f}\cdot\beta &= \delta\cdot\mathrm{d}_{\mathrm{N}} - \mathrm{d}_{\mathrm{M}}\cdot\delta \equiv [\delta, \mathrm{d}]
\end{aligned}$$

The first equation says that $\mathrm{g} : \mathrm{N} \to \mathrm{M} \in \mathrm{Z}^0\mathrm{dgMod}\text{-}\mathcal{C}$. The second and the third say that g is homotopy inverse to f.

The equation $\mathrm{d}_{\mathsf{Cone}\,\mathrm{f}}\mathrm{h} + \mathrm{h}\mathrm{d}_{\mathsf{Cone}\,\mathrm{f}} = 1_{\mathsf{Cone}\,\mathrm{f}}$ can be written as the system

$$\begin{aligned} \mathrm{d_N g} &= \mathrm{g d_M}, \\ 1_\mathrm{M} - \mathrm{f} \cdot \mathrm{g} &= \mathrm{d_M}\alpha + \alpha\mathrm{d_M}, \\ 1_\mathrm{N} - \mathrm{g} \cdot \mathrm{f} &= \mathrm{d_N}\beta + \beta\mathrm{d_N}, \\ \alpha \cdot \mathrm{f} - \mathrm{f} \cdot \beta &= \delta \cdot \mathrm{d_N} - \mathrm{d_M} \cdot \delta \equiv [\delta, \mathrm{d}] \end{aligned}$$

The first equation says that $\mathrm{g} : \mathrm{N} \to \mathrm{M} \in \mathrm{Z}^0\mathrm{dgMod}\text{-}\mathcal{C}$. The second and the third say that g is homotopy inverse to f.

The fourth equation says that we deal with representation of Kontsevich's category!

Assume that $f : M \to N \in Z^0\mathrm{dgMod}\text{-}\mathcal{C}$ is homotopy invertible. The category $H^0\mathrm{dgMod}\text{-}\mathcal{C}$ is triangulated. The square

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & N \arrow[d, "1"] \\ N \arrow[r, "1"] & N \end{tikzcd}$$

extends to a morphism of distinguished triangles

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "f"'] & N \arrow[r] \arrow[d, "1"] & \mathsf{Cone}\, f \arrow[r] \arrow[d, "0"] & M[1] \arrow[d, "f[1]"] \\ N \arrow[r, "1"] & N \arrow[r] & 0 \arrow[r] & N[1] \end{tikzcd}$$

by property [TR3] of triangulated category $H^0\mathrm{dgMod}\text{-}\mathcal{C}$.

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$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "f"'] & N \arrow[r] \arrow[d, "1"] & \mathsf{Cone}\, f \arrow[r] \arrow[d, "0"] & M[1] \arrow[d, "f[1]"] \\ N \arrow[r, "1"] & N \arrow[r] & 0 \arrow[r] & N[1] \end{tikzcd}$$

by property [TR3] of triangulated category $H^0\mathrm{dgMod}\text{-}\mathcal{C}$. The morphism $0 : \mathsf{Cone}\, f \to 0$ is invertible in $H^0\mathrm{dgMod}\text{-}\mathcal{C}$, that is, $\mathsf{Cone}\, f$ is contractible. □

Квазі-ізоморфізм гомотопійно проективних модулів гомотопійно оборотний

Proposition

Let $f : M \to N$ be a quasi-isomorphism of homotopy projective $\mathcal{C}$-modules. Then f is homotopy invertible.

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Let $\mathrm{f} : \mathrm{M} \to \mathrm{N}$ be a quasi-isomorphism of homotopy projective $\mathcal{C}$-modules. Then f is homotopy invertible.

Доведення.

Since f is a quasi-isomorphism, $\mathsf{Cone}\,\mathrm{f}$ is acyclic by Lemma 1. On the other hand, $\mathsf{Cone}\,\mathrm{f}$ is homotopy projective. The identity morphism $\mathsf{id} : \mathsf{Cone}\,\mathrm{f} \to \mathsf{Cone}\,\mathrm{f} \in \mathrm{Z}^0\mathrm{dgMod}\text{-}\mathcal{C}$ is a morphism from a homotopy projective $\mathcal{C}$-module to an acyclic $\mathcal{C}$-module. Hence, $[\mathsf{id}_{\mathsf{Cone}\,\mathrm{f}}] = 0 \in \mathrm{H}^0\mathrm{dgMod}\text{-}\mathcal{C}$, that is, $\mathsf{Cone}\,\mathrm{f}$ is contractible. By Lemma 2 f is homotopy invertible. $\square$

Lemma (see Verdier, Proposition 2.3.3)

Consider a diagram in $Z^0\mathrm{dgMod}\text{-}\mathcal{C}$

$$\begin{tikzcd} P \arrow[dr, "f"] & \\ Q \arrow[r, "r", "\sim"'] & N \end{tikzcd}$$

where r is a quasi-isomorphism, $P \in \text{h-proj-}\mathcal{C}$.

Lemma (see Verdier, Proposition 2.3.3)

Consider a diagram in $Z^0\mathrm{dgMod}\text{-}\mathcal{C}$

$$\begin{tikzcd} P \arrow[drr, "f"] & & \\ Q \arrow[rr, "r", "\sim"'] & & N \end{tikzcd}$$

where r is a quasi-isomorphism, $P \in \text{h-proj-}\mathcal{C}$.
Then there is a morphism $g : P \to Q \in Z^0\mathrm{dgMod}\text{-}\mathcal{C}$ such that the triangle

$$\begin{tikzcd} P \arrow[d, "g"'] \arrow[drr, "f"] \arrow[dr, phantom, "\sim"] & & \\ Q \arrow[rr, "r", "\sim"'] & & N \end{tikzcd} \tag{1}$$

commutes up to homotopy.

Доведення.

$\mathsf{Cone}\,\mathrm{r}$ is acyclic by Lemma 1. The composition

$$\mathrm{P} \xrightarrow{\mathrm{f}} \mathrm{N} \xhookrightarrow{\alpha(\mathrm{r})=(0\ 1)} \mathsf{Cone}(\mathrm{r} : \mathrm{Q} \to \mathrm{N}) = (\mathrm{Q}[1] \oplus \mathrm{N}, \mathrm{d}_{\mathsf{Cone}\,\mathrm{r}})$$

is null-homotopic, being a map from a homotopy projective module to an acyclic module. Therefore, there is a homotopy $\mathrm{h} \in \mathrm{dgMod}\text{-}\mathcal{C}(\mathrm{P}, (\mathrm{Q}[1] \oplus \mathrm{N}, \mathrm{d}_{\mathsf{Cone}\,\mathrm{r}}))^{-1}$ such that

$$\mathrm{f} \cdot \alpha(\mathrm{r}) = \mathrm{d}_{\mathrm{P}} \cdot \mathrm{h} + \mathrm{h} \cdot \mathrm{d}_{\mathsf{Cone}\,\mathrm{r}}. \tag{2}$$

Доведення.

$\mathsf{Cone}\,r$ is acyclic by Lemma 1. The composition

$$P \xrightarrow{f} N \xhookrightarrow{\alpha(r)=(0\ 1)} \mathsf{Cone}(r : Q \to N) = (Q[1] \oplus N, d_{\mathsf{Cone}\,r})$$

is null-homotopic, being a map from a homotopy projective module to an acyclic module. Therefore, there is a homotopy $h \in \mathrm{dgMod}\text{-}\mathcal{C}(P, (Q[1] \oplus N, d_{\mathsf{Cone}\,r}))^{-1}$ such that

$$f \cdot \alpha(r) = d_P \cdot h + h \cdot d_{\mathsf{Cone}\,r}. \tag{2}$$

Representing the homotopy as $h = (g \cdot \sigma, \underline{\gamma})$, $g \in \mathrm{dgMod}\text{-}\mathcal{C}(Q, Q)^0$, $\gamma \in \mathrm{dgMod}\text{-}\mathcal{C}(Q, N)^0$ we see that (2) is equivalent to requirements

$$g \in Z^0\mathrm{dgMod}\text{-}\mathcal{C}(P, Q),$$
$$f = g \cdot r + d_P \cdot \gamma + \gamma \cdot d_N,$$

which is the precise statement about homotopy commutativity of (1). □

Corollary

(a) Given a diagram in $\mathrm{Z}^0\mathrm{dgMod}\text{-}\mathcal{C}$

$$\begin{tikzcd} \mathrm{P} \arrow[d, "\mathrm{t}"'] & \mathrm{Q} \arrow[d, "\mathrm{r}", "\wr"'] \\ \mathrm{M} \arrow[r, "\mathrm{j}"] & \mathrm{N} \end{tikzcd}$$

with homotopy projective P and a quasi-isomorphism r there is a morphism $\mathrm{g} : \mathrm{P} \to \mathrm{Q} \in \mathrm{Z}^0\mathrm{dgMod}\text{-}\mathcal{C}$ such that the square

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commutes up to homotopy.

Corollary

(a) Given a diagram in $Z^0\mathrm{dgMod}\text{-}\mathcal{C}$

$$\begin{tikzcd} P \arrow[d, "t"'] & Q \arrow[d, "r", "\wr"'] \\ M \arrow[r, "j"] & N \end{tikzcd}$$

with homotopy projective P and a quasi-isomorphism r there is a morphism $g : P \to Q \in Z^0\mathrm{dgMod}\text{-}\mathcal{C}$ such that the square

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "t"'] \arrow[dr, phantom, "\sim"] & Q \arrow[d, "r", "\wr"'] \\ M \arrow[r, "j"] & N \end{tikzcd}$$

commutes up to homotopy.

(b) If, moreover, Q is homotopy projective and t, j are quasi-isomorphisms, then g is homotopy invertible.

Доведення.

(b) The left hand side of equation $t \cdot j - d_P \cdot \gamma - \gamma \cdot d_N = g \cdot r$ is a quasi-isomorphism, since $H^\bullet(t \cdot j - d_P \cdot \gamma - \gamma \cdot d_N) = H^\bullet(t \cdot j)$. Therefore, g is a quasi-isomorphism. By Proposition 1 g is homotopy invertible. $\square$

Доведення.

(b) The left hand side of equation $\mathrm{t} \cdot \mathrm{j} - \mathrm{d_P} \cdot \gamma - \gamma \cdot \mathrm{d_N} = \mathrm{g} \cdot \mathrm{r}$ is a quasi-isomorphism, since $\mathrm{H}^\bullet(\mathrm{t} \cdot \mathrm{j} - \mathrm{d_P} \cdot \gamma - \gamma \cdot \mathrm{d_N}) = \mathrm{H}^\bullet(\mathrm{t} \cdot \mathrm{j})$. Therefore, g is a quasi-isomorphism. By Proposition 1 g is homotopy invertible. $\square$

2.3.2. Soient $F : \mathcal{C} \longrightarrow \mathcal{C}'$ un foncteur et X un objet de $\mathcal{C}'$. Rappelons qu'un objet Y de $\mathcal{C}$ est dit *F-libre à droite sur X*, ou simplement *libre à droite sur X*, s'il existe un morphisme $u : F(Y) \longrightarrow X$ tel que pour tout morphisme $F(Z) \xrightarrow{v} X$, il existe un et un seul morphisme $w : Z \longrightarrow Y$ tel que le diagramme ci-après soit commutatif :

$$\begin{tikzcd}
F(Y) \arrow[rr, "u"] & & X \\
& F(Z) \arrow[ul, "F(w)"] \arrow[ur, "v"'] &
\end{tikzcd}$$

.

Об'єкти вільні зліва

La flèche $u : F(Y) \longrightarrow X$ s'appelle *flèche de liberté* pour le couple (X, Y) (relativement au foncteur F). Celà peut encore s'interpréter, en prenant un univers dont $\mathcal{C}$ et $\mathcal{C}'$ soient des éléments, en disant que Y est libre à droite sur X s'il existe un isomorphisme entre les foncteurs $Z \mapsto \mathsf{Hom}_{\mathcal{C}}(Z, Y)$ et $Z \mapsto \mathsf{Hom}_{\mathcal{C}'}(F(Z), X)$. Nous dirons qu'un objet Y de $\mathcal{C}$ est *F-libre à droite* s'il existe un objet X de $\mathcal{C}'$ tel que Y soit libre à droite sur X. Nous dirons qu'un objet X de $\mathcal{C}'$ est *F-libérable à droite* s'il existe un objet Y de $\mathcal{C}$ libre à droite sur X. On définit de façon analogue, par passage aux catégories opposées, les notions d'objet libre ou libérable *à gauche*.

Властивості гомотопійно проективних модулів

Proposition 2.3.3. *Soient $\mathcal{D}$ une catégorie triangulée, $\mathcal{B}$ une sous-catégorie triangulée pleine, $Q\ :\mathcal{D} \longrightarrow \mathcal{D}/\mathcal{B}$ le foncteur de passage au quotient* (2.2.10), *$S_{\mathcal{D}}(\mathcal{B})$ le système multiplicatif de $\mathcal{D}$ associé à $\mathcal{B}$* (2.1.8).

a) *Pour tout objet Y de $\mathcal{D}$, les conditions suivantes sont équivalentes :*

i) *L'objet Y est Q-libre à droite.*

ii) *Pour tout morphisme $s \in S_{\mathcal{D}}(\mathcal{B})$, $s : S \longrightarrow T$, l'application :*

$$\mathsf{Hom}_{\mathcal{D}}(s,Y) : \mathsf{Hom}_{\mathcal{D}}(T,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(S,Y)$$

est une bijection.

iii) *Pour tout objet B de $\mathcal{B}$, $\mathsf{Hom}_{\mathcal{D}}(B,Y) = 0$.*

iv) *Tout morphisme $Y \xrightarrow{s} Z$, avec $s \in S_{\mathcal{D}}(\mathcal{B})$, admet une rétraction.*

v) *Pour tout objet X de $\mathcal{D}$, l'application :*

$$\mathsf{Hom}_{\mathcal{D}}(X,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}/\mathcal{B}}(Q(X),Q(Y))$$

est bijective.

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$$\begin{tikzcd} & P \arrow[dl] \arrow[d, equal] \\ Z \arrow[r, "\sim"] & P \end{tikzcd}$$

est une bijection.

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$$\mathsf{Hom}_{\mathcal{D}}(X, Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}/\mathcal{B}}(Q(X), Q(Y))$$

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$\mathcal{D}/\mathcal{B} = \mathcal{D}(\mathcal{C})^{op}$

Для нас $\mathcal{D} = \mathrm{H}^0\mathrm{dgMod}\text{-}\mathcal{C}^{\mathsf{op}}$, $\mathcal{B} = \mathrm{H}^0\mathrm{dgAcyc}\text{-}\mathcal{C}^{\mathsf{op}}$, $\mathrm{S}_{\mathcal{D}}(\mathcal{B}) = \{\mathrm{qis}\}^{\mathsf{op}}$.

2.3.4. Démonstration de la proposition 2.3.3 : Démontrons a). Tout d'abord, si Y est Q-libre à droite, le foncteur $X \mapsto \mathsf{Hom}_{\mathcal{D}}(X,Y)$ est isomorphe au foncteur $X \mapsto \mathsf{Hom}_{\mathcal{D}/\mathcal{B}}(Q(X),Z)$, pour un objet Z convenable de $\mathcal{D}/\mathcal{B}$. Par suite, le foncteur $X \mapsto \mathsf{Hom}_{\mathcal{D}}(X,Y)$ transforme les morphismes de $S_{\mathcal{D}}(\mathcal{B})$ en isomorphismes. Ceci démontre l'implication $i) \Rightarrow ii)$. L'implication $ii) \Rightarrow iii)$ résulte de ce que le morphisme $B \longrightarrow 0$ est un morphisme de $S_{\mathcal{D}}(\mathcal{B})$. L'implication $iii) \Rightarrow iv)$ résulte de la définition de $S_{\mathcal{D}}(\mathcal{B})$ et de (1.2.6). L'implication $iv) \Rightarrow v)$ résulte de l'isomorphisme (2.2.4) :

$$\mathsf{Hom}_{\mathcal{D}/\mathcal{B}}(Q(X),Q(Y)) \xrightarrow{\sim} \varinjlim_{Y\backslash S_{\mathcal{D}}(\mathcal{B})} \mathsf{Hom}_{\mathcal{D}}(X,\,.\,) \quad .$$

Enfin, l'implication $v) \Rightarrow i)$ est évidente : Y est libre à droite sur $Q(Y)$.

Кодобутки і добутки в похідній категорії

Lemma

Let $\mathcal{A}$ be a small abelian category and let $\{A^*_n\}_{n\geq 0} \subseteq \mathsf{Ob}(D^?(\mathcal{A}))$.

(1) If $A^i_n = 0$ for all $i > -n$ and $? = -, \emptyset$, then $\bigoplus_{n=0}^{\infty} A^*_n$ is a coproduct in $D^?(\mathcal{A})$, i.e. there is a canonical isomorphism

$$\bigoplus_{n=0}^{\infty} A^*_n \cong \coprod_{n=0}^{\infty} A^*_n.$$

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Corollary

Let $\mathcal{A}$ be a small abelian category and let $\{A^i\}_{i\geq 0} \subseteq \mathsf{Ob}(\mathcal{A})$. Then $\bigoplus_{i=0}^{\infty} A^i[k+i]$ is a coproduct in $D^-(\mathcal{A})$ and $D(\mathcal{A})$, while $\bigoplus_{i=0}^{\infty} A^i[k-i]$ is a product in $D^+(\mathcal{A})$ and $D(\mathcal{A})$, for all $k \in \mathbb{Z}$.

Alberto Canonaco, Amnon Neeman, and Paolo Stellari, Uniqueness of enhancements for derived and geometric categories, 2021, arXiv:2101.04404 §1.5 – §1.6

Jean-Louis Verdier, Des catégories dérivées des catégories abéliennes, Astérisque (1996), no. 239, xii+253 pp., With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. §2.3.2 – §2.3.4