7. Породження триангульованих категорій. Навколо похідних категорій

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Let us recall that when $\mathcal{A}$ is a small additive category, then $\mathrm{K}(\mathcal{A})$ denotes the homotopy category of complexes. Namely, its objects are cochain complexes of objects in $\mathcal{A}$, while its morphisms are homotopy equivalence classes of morphisms of complexes. For $\mathrm{A}^* \in \mathsf{Ob}(\mathrm{K}(\mathcal{A}))$, we denote by A^i its i-th component. We can then define the full subcategories $\mathrm{K}^\mathrm{b}(\mathcal{A})$, $\mathrm{K}^+(\mathcal{A})$, $\mathrm{K}^-(\mathcal{A})$ of the category $\mathrm{K}(\mathcal{A})$ whose objects are

$$\mathsf{Ob}(\mathrm{K}^\mathrm{b}(\mathcal{A})) = \left\{\mathrm{A}^* \in \mathrm{K}(\mathcal{A}) \mid \mathrm{A}^\mathrm{i} = 0 \text{ for all } |\mathrm{i}| \gg 0\right\}$$

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$$\mathsf{Ob}(\mathrm{K}^{-}(\mathcal{A})) = \left\{\mathrm{A}^* \in \mathrm{K}(\mathcal{A}) \mid \mathrm{A}^{\mathrm{i}} = 0 \text{ for all } \mathrm{i} \gg 0\right\}$$

For $\mathsf{?} = \mathrm{b}, +, -, \emptyset$, we single out the full subcategory $\mathrm{V}^{\mathsf{?}}(\mathcal{A}) \subseteq \mathrm{K}^{\mathsf{?}}(\mathcal{A})$ consisting of objects with zero differentials. It will be crucial in the rest. Here we just point out that, for an object $\mathrm{A}^* \in \mathrm{V}^{\mathsf{?}}(\mathcal{A})$, we will use the shorthand

$$\bigoplus_{\mathrm{i} \in \mathbb{Z}} \mathrm{A}^{\mathrm{i}}[-\mathrm{i}]$$

to remind that the object $\mathrm{A}^{\mathrm{i}} \in \mathcal{A}$ is placed in degree i.

When $\mathcal{A}$ is an abelian category, the full triangulated subcategory $\mathrm{K}^?_{\mathrm{acy}}(\mathcal{A}) \subseteq \mathrm{K}^?(\mathcal{A})$ consists of acyclic complexes, i.e. objects in $\mathrm{K}(\mathcal{A})$ with trivial cohomology.

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The triangulated category $\mathrm{D}^?(\mathcal{A})$ is then the Verdier quotient of $\mathrm{K}^?(\mathcal{A})$ by $\mathrm{K}^?_{\mathrm{acy}}(\mathcal{A})$, and it comes with a quotient functor

$$\mathrm{Q} : \mathrm{K}^?(\mathcal{A}) \longrightarrow \mathrm{D}^?(\mathcal{A}).$$

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We can then consider the full subcategory $\mathrm{B}^?(\mathcal{A}) \subseteq \mathrm{D}^?(\mathcal{A})$ as

$$\mathrm{B}^?(\mathcal{A}) := \mathrm{Q}(\mathrm{V}^?(\mathcal{A})).$$

Породження триангульованих категорій

Definition

Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S} \subset \mathsf{Ob}(\mathcal{T})$. We define

1. $\langle \mathcal{S} \rangle_1$ is the collection of all direct summands of finite coproducts of shifts of objects in $\mathcal{S}$;

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2. $\langle \mathcal{S} \rangle_{n+1}$ consists of all direct summands of objects $T \in \mathcal{T}$, for which there exists a distinguished triangle $T_1 \to T \to T_2$ with $T_1 \in \langle \mathcal{S} \rangle_n$ and $T_2 \in \langle \mathcal{S} \rangle_1$.

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We set $\langle \mathcal{S} \rangle_\infty$ for the full subcategory consisting of all objects T in $\mathcal{T}$ contained in $\langle \mathcal{S} \rangle_n$, for some n.

Proposition

Recall $\mathrm{V}^?(\mathcal{A}) \subset \mathrm{K}^?(\mathcal{A})$. For $? = \mathrm{b}, +, -, \emptyset$, we have that $\left\langle \mathrm{V}^?(\mathcal{A}) \right\rangle_3 = \mathrm{K}^?(\mathcal{A})$.

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Доведення. Let $\mathrm{A}^* \in \mathsf{Ob}(\mathrm{K}^?(\mathcal{A}))$, which we write as a complex

$$\cdots \longrightarrow \mathrm{A}^{-2} \longrightarrow \mathrm{A}^{-1} \longrightarrow \mathrm{A}^{0} \longrightarrow \mathrm{A}^{1} \longrightarrow \mathrm{A}^{2} \longrightarrow \cdots$$

Let K^{i} be the kernel of the differential $\mathrm{A}^{\mathrm{i}} \to \mathrm{A}^{\mathrm{i}+1}$. Then the map $\mathrm{A}^{\mathrm{i}-1} \to \mathrm{A}^{\mathrm{i}}$ factors uniquely as $\mathrm{A}^{\mathrm{i}-1} \xrightarrow{\alpha^{\mathrm{i}}} \mathrm{K}^{\mathrm{i}} \hookrightarrow \mathrm{A}^{\mathrm{i}}$.

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This yields the morphism

$$\bigoplus_{\mathrm{i}\in\mathbb{Z}} \mathrm{A}^{\mathrm{i}-1}[-\mathrm{i}] \xrightarrow{\bigoplus_{\mathrm{i}\in\mathbb{Z}} \alpha^\mathrm{i}} \bigoplus_{\mathrm{i}\in\mathbb{Z}} \mathrm{K}^\mathrm{i}[-\mathrm{i}]$$

in $\mathrm{V}^?(\mathcal{A})$. Denote by C^* its mapping cone. It is clear that $\mathrm{C}^* \in \left\langle \mathrm{V}^?(\mathcal{A}) \right\rangle_2$ and it is the direct sum over $\mathrm{i} \in \mathbb{Z}$ of the complexes

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{A}^{\mathrm{i}-1} \xrightarrow{\alpha^\mathrm{i}} \mathrm{K}^\mathrm{i} \longrightarrow 0 \longrightarrow \cdots$$

Now consider the cochain map

$$\bigoplus_{\mathrm{i}\in\mathbb{Z}} \mathrm{K}^{\mathrm{i}}[-\mathrm{i}] \xrightarrow{\quad \varphi+\psi \quad} \mathrm{C}^*$$

whose components, out of $\mathrm{K}^{\mathrm{i}}[-\mathrm{i}]$, are (respectively) φ^{i} as below

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{K}^{\mathrm{i}} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{A}^{\mathrm{i}-1} & \longrightarrow & \mathrm{K}^{\mathrm{i}} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Now consider the cochain map

$$\bigoplus_{i\in\mathbb{Z}} K^i[-i] \xrightarrow{\varphi+\psi} C^*$$

whose components, out of $K^i[-i]$, are (respectively) φ^i as below

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 & & \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & A^{i-1} & \longrightarrow & K^i & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
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 & & \downarrow & & \downarrow & & \hookdownarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A^i & \longrightarrow & K^{i+1} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Породження похідної категорії в 3 кроки

It can be easily checked that the mapping cone of the morphism $\varphi + \psi$ is isomorphic to the direct sum of the complex A^* and of complexes of the form

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In other words, $\mathrm{Cone}\,(\varphi + \psi) \cong \mathrm{A}^*$ in $\mathrm{K}^?(\mathcal{A})$. Therefore, as C^* belongs to $\left\langle \mathrm{V}^?(\mathcal{A}) \right\rangle_2$ and $\varphi + \psi$ is a morphism from an object of $\mathrm{V}^?(\mathcal{A})$ to C^*, we have that $\mathrm{A}^* \in \left\langle \mathrm{V}^?(\mathcal{A}) \right\rangle_3$. □

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Corollary

Recall $\mathrm{B}^?(\mathcal{A}) \subset \mathrm{D}^?(\mathcal{A})$. For $? = \mathrm{b}, +, -, \emptyset$, we have that $\left\langle \mathrm{B}^?(\mathcal{A}) \right\rangle_3 = \mathrm{D}^?(\mathcal{A})$.

$\mathrm{Cone}\,(\varphi+\psi)^{\mathrm{i}} = (\mathrm{K}^{\mathrm{i}+1} \oplus \mathrm{A}^{\mathrm{i}} \oplus \mathrm{K}^{\mathrm{i}}, \mathrm{d})$ has the differential $\mathrm{d} = \begin{pmatrix} 0 & \iota^{\mathrm{i}+1} & 1 \\ 0 & 0 & \alpha^{\mathrm{i}+1} \\ 0 & 0 & 0 \end{pmatrix}$. We have

$$\left(\mathrm{A}^{\mathrm{i}-1} \xrightarrow{\alpha^{\mathrm{i}}} \mathrm{K}^{\mathrm{i}} \xhookrightarrow{\iota^{\mathrm{i}}} \mathrm{A}^{\mathrm{i}}\right) = \mathrm{d}, \qquad \left(\mathrm{K}^{\mathrm{i}} \xhookrightarrow{\iota^{\mathrm{i}}} \mathrm{A}^{\mathrm{i}} \xrightarrow{\alpha^{\mathrm{i}+1}} \mathrm{K}^{\mathrm{i}+1}\right) = 0.$$

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There is a chain map

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\begin{pmatrix} 1 & 0 & 0 \\ \alpha^{\mathrm{i}+1} & 1 & 0 \\ 0 & -\iota^{\mathrm{i}} & 1 \end{pmatrix}\Big\downarrow & & \Big\downarrow\begin{pmatrix} 1 & 0 & 0 \\ \alpha^{\mathrm{i}+2} & 1 & 0 \\ 0 & -\iota^{\mathrm{i}+1} & 1 \end{pmatrix} \\
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It is invertible $\Rightarrow \mathrm{Cone}\,(\varphi+\psi) \cong ((\mathrm{A}^{\mathrm{i}}), -\mathrm{d}) \in \mathrm{K}^{?}(\mathcal{A})$.

Модельна структура на dg$\mathcal{C}$at

If we consider for $\mathcal{C}$ the category **DCAT**, *for W the subcategory of quasi-equivalences, the category* **DCAT** *admits a Quillen model structure whose weak equivalences are the quasi-equivalences.*

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$\mathsf{dg}\mathcal{C}\mathsf{at}$ has a model structure whose weak equivalences are quasi-equivalences and such that every object is fibrant. We denote by Hqe the corresponding homotopy category, namely the localization of $\mathsf{dg}\mathcal{C}\mathsf{at}$ with respect to quasi-equivalences. Since H^0 sends quasi-equivalences to equivalences, for every morphism $\mathrm{f}\colon \mathcal{C}_1 \to \mathcal{C}_2$ in Hqe there is a $\Bbbk$-linear functor $\mathrm{H}^0(\mathrm{f})\colon \mathrm{H}^0(\mathcal{C}_1) \to \mathrm{H}^0(\mathcal{C}_2)$, which is well-defined up to equivalences.

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Dg functors between two dg categories $\mathcal{C}_1$ and $\mathcal{C}_2$ form in a natural way the objects of a dg category $\underline{\mathrm{Hom}}(\mathcal{C}_1, \mathcal{C}_2)$. For every dg category $\mathcal{C}$ we set $\mathrm{dgMod}(\mathcal{C}) := \underline{\mathrm{Hom}}(\mathcal{C}^{\mathsf{op}}, \mathrm{C}_{\mathsf{dg}}(\mathrm{Mod}(\Bbbk)))$ and call its objects (right) dg $\mathcal{C}$-modules.

Зсуви в dg-категорії

Given an object $A \in \mathcal{A}$, the object $A[r]$ is characterized (up to a DG isomorphism) by the existence of closed morphisms $f : A \to A[r]$, $g : A[r] \to A$ of degrees $-r$ and r, respectively, such that $fg = gf = 1$. Thus, in particular, every DG functor commutes with shifts.

Definition 4.6. The objects of $\mathcal{A}^{\text{pre-tr}}$ are "one-sided twisted complexes," that is, formal expressions $(\oplus_{i=1}^{n} C_i[r_i], q)$, where $C_i \in \operatorname{Ob}\mathcal{A}$, $r_i \in \mathbb{Z}$, $n \geq 0$, $q = (q_{ij})$, $q_{ij} \in \operatorname{Hom}(C_j[r_j], C_i[r_i])$ is homogeneous of degree 1, $q_{ij} = 0$ for $i \geq j$, and $dq + q^2 = 0$. If $C, C' \in \operatorname{Ob}\mathcal{A}^{\text{pre-tr}}$, $C = (\oplus C_j[r_j], q)$, $C' = (\oplus C'_j[r'_j], q')$, then the $\mathbb{Z}$-graded k-module $\operatorname{Hom}(C, C')$ is the space of matrices $f = (f_{ij})$, $f_{ij} \in \operatorname{Hom}(C_j[r_j], C'_i[r'_i])$, and the composition map $\operatorname{Hom}(C, C') \otimes \operatorname{Hom}(C', C'') \to \operatorname{Hom}(C, C'')$ is matrix multiplication. The differential $d : \operatorname{Hom}(C, C') \to \operatorname{Hom}(C, C')$ is defined by $df := (df_{ij}) + q'f - (-1)^l fq$ if $\deg f_{ij} = l$.

Notice that the DG category $\mathcal{A}^{\text{pre-tr}}$ is closed under formal shifts:

$$\left(\oplus_{i=1}^{n} C_i[r_i], q\right)[1] = \left(\oplus_{i=1}^{n} C_i[r_i + 1], -q\right). \tag{4.15}$$

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$$A \xrightarrow{f} B \xrightarrow{g} A, \quad fg = 1 = gf, \quad df = 0 = dg, \quad \deg f = -r, \ \deg g = r,$$

$$A \xrightarrow{f'} C \xrightarrow{g'} A, \ f'g' = 1 = g'f', \ df' = 0 = dg', \ \deg f' = -r, \ \deg g' = r,$$

$$k = (B \xrightarrow{g} A \xrightarrow{f'} C), j = (C \xrightarrow{g'} A \xrightarrow{f} B), k, j \in Z^0\mathcal{A}, j = k^{-1} \Rightarrow B \cong C$$

Конус в dg-категорії

Definition 4.7. Let $\mathcal{B}$ be a DG category and let $f \in \mathrm{Hom}(A, B)$ be a closed degree-zero morphism in $\mathcal{B}$. An object $C \in \mathcal{B}$ is called the cone of f, denoted $\mathrm{Cone}(f)$, if $\mathcal{B}$ contains the object $A[1]$ and there exist degree-zero morphisms

$$A[1] \xrightarrow{i} C \xrightarrow{p} A[1], \qquad B \xrightarrow{j} C \xrightarrow{s} B, \tag{4.16}$$

with the properties

$$pi = 1, \qquad sj = 1, \qquad si = 0, \qquad pj = 0, \qquad ip + js = 1, \tag{4.17}$$

and

$$d(j) = d(p) = 0, \qquad d(i) = jf, \qquad d(s) = -fp. \tag{4.18}$$

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$$di = j \circ f \circ \sigma^{-1}, \qquad ds = -f \circ \sigma^{-1} \circ p,$$

where

$$\sigma : A \to A[1], \quad \deg \sigma = -1, \qquad \sigma^{-1} : A[1] \to A, \quad \deg \sigma^{-1} = 1.$$

Lemma 4.8. The cone of a closed degree-zero morphism is uniquely defined up to a DG isomorphism. □

Proof. Note that the first set of conditions means that C is the direct sum of $A[1]$ and B in the corresponding graded category $\mathcal{B}^{gr}$. Thus for any object E in $\mathcal{A}$, there are isomorphisms of graded k-modules

$$\begin{aligned} \mathrm{Hom}(E, C) &= \mathrm{Hom}\left(E, A[1]\right) \oplus \mathrm{Hom}(E, B), \\ \mathrm{Hom}(C, E) &= \mathrm{Hom}\left(A[1], E\right) \oplus \mathrm{Hom}(B, E), \end{aligned} \tag{4.19}$$

which are given by composing with i and j (or with p and s). Then the second set of conditions determines the differentials in $\mathrm{Hom}(E, C)$ and $\mathrm{Hom}(C, E)$. ■

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We have a right dg-module $M : \mathcal{B}^{\mathsf{op}} \to \mathsf{dg}$,
$E \mapsto (\mathcal{B}(E, A[1]) \oplus \mathcal{B}(E, B), d_M)$, where d_M comes from the decomposition $g = i \circ (p \circ g) + j \circ (s \circ g) : E \to C$, namely,

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$$d_M = \begin{pmatrix} d & 0 \\ f \circ \sigma^{-1} \circ ? & d \end{pmatrix} \qquad \text{– left matrix.}$$

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$M \cong \mathcal{B}(E,C)$ is representable (by C) iff $\exists\, \mathsf{Cone}\, f \in \mathcal{B}$ $(= C)$.

dg-вкладення Йонеда

Lemma 7.3.5 ($\mathcal{V}$-Yoneda lemma). *Given a small $\mathcal{V}$-category $\underline{\mathcal{D}}$, and object $d \in \underline{\mathcal{D}}$, and a $\mathcal{V}$-functor $F : \underline{\mathcal{D}} \to \underline{\mathcal{V}}$, the canonical map is a $\mathcal{V}$-natural isomorphism*

$$Fd \xrightarrow{\cong} \underline{\mathcal{V}}^{\underline{\mathcal{D}}}(\underline{\mathcal{D}}(d, -), F).$$

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Corollary

For every dg category $\mathcal{C}$ the map defined on objects by $A \mapsto \mathcal{C}(-, A)$ extends to a fully faithful dg functor $\mathsf{Y}^{\mathcal{C}}_{\mathrm{dg}} : \mathcal{C} \to \mathrm{dgMod}(\mathcal{C})$ (the dg Yoneda embedding). ~~It is easy to see that the image of~~

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We now give a definition of representable functor in the present situation. Let E be an object in the DG-category $\mathscr{A}$. It determines a contravariant DG-functor $h_E : \mathscr{A} \to C(\mathscr{A} b)$ that takes $F \in \mathrm{Ob}\,\mathscr{A}$ into the complex $\mathrm{Hom}^{\boldsymbol{\cdot}}_{\mathscr{A}}(F, E)$. The assignment $E \mapsto h_E$ gives a covariant DG-functor

$$h : \mathscr{A} \to DG\text{-}\mathrm{Fun}^0(\mathscr{A}, C(\mathscr{A} b)).$$

As in the "classical" case (see [18]), one verifies that the functor h is fully strict, i.e., that there exist isomorphisms of complexes

$$\mathrm{Hom}^{\boldsymbol{\cdot}}_{\mathscr{A}}(E, E') \simeq \mathrm{Hom}_{DG\text{-}\mathrm{Fun}^0(\mathscr{A}, C(\mathscr{A} b))}(h_E, h_{E'}). \qquad (1.3)$$

A contravariant DG-functor $h : \mathscr{A} \to C(\mathscr{A} b)$ will be called *representable* if it is isomorphic (as a DG-functor) to a functor of the form h_E for some $E \in \mathrm{Ob}\,\mathscr{A}$.

Вкладення Pre-Tr($\mathcal{A}$) в dgMod($\mathcal{A}$)

DEFINITION 3. Let $\mathscr{A}$ be a DG-category. We define an imbedding of DG-categories

$$\alpha\colon \text{Pre-Tr}(\mathscr{A}) \to DG\text{-Fun}^0(\mathscr{A}, C(\mathscr{A}b)).$$

The imbedding assigns to an object $K = \{E_i, q_{ij}\} \in \text{Ob}\,\text{Pre-Tr}(\mathscr{A})$ the following DG-functor $\alpha(K)\colon \mathscr{A} \to C(\mathscr{A}b)$. For each $E \in \text{Ob}\,\mathscr{A}$ the value $\alpha(K)(E)$ is the graded abelian group $\bigoplus \text{Hom}_{\mathscr{A}}(E, E_i)[n_i]$ provided with the differential $d+Q$, where $Q = \|q_{ij}\|$ and d is the differential in $\bigoplus \text{Hom}_{\mathscr{A}}(E, E_i)[n_i]$.

PROPOSITION 3. (a) *The functor* α *is an imbedding of* Pre-Tr($\mathscr{A}$) *into* DG-Fun$^0(\mathscr{A}, C(\mathscr{A}b))$ *as a full* DG*-subcategory, and it takes the cone of a closed morphism* f *in* Pre-Tr($\mathscr{A}$) *into the cone of the morphism* $\alpha(f)$ *in* DG-Fun$^0(\mathscr{A}, C(\mathscr{A}b))$.

Вкладення $\mathsf{Pre\text{-}Tr}(\mathcal{A})$ в $\mathrm{dgMod}(\mathcal{A})$

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Доведення.

Morphisms in $\mathsf{Pre\text{-}Tr}(\mathcal{A})$ are given by rectangular matrix with entries in $\mathcal{A}(\mathrm{E_i}, \mathrm{E'_j})[\mathrm{n'_j} - \mathrm{n_i}]$. Morphisms in $\mathrm{dgMod}(\mathcal{A})$ are given by rectangular matrix of the same size with entries in $\mathrm{dgMod}(\mathcal{A})(\mathcal{A}(\_, \mathrm{E_i})[\mathrm{n_i}], \mathcal{A}(\_, \mathrm{E'_j})[\mathrm{n'_j}])$. Differentials agree. □

Трикутники в dg-категорії з конусами

Given a closed degree-zero morphism $f : A \to B$, the diagram

$$A \xrightarrow{f} B \xrightarrow{j} \mathrm{Cone}(f) \xrightarrow{p} A[1] \tag{4.20}$$

is called a pre-exact triangle.

Remark 4.9. It is clear that any DG functor preserves cones of closed degree-zero morphisms and preserves pre-exact triangles.

Замкненість $\mathcal{A}^{\text{pre-tr}}$ стосовно конусів

Proposition 4.10 [5]. Let $\mathcal{A}$ be a DG category. Then

(a) the DG category $\mathcal{A}^{\text{pre-tr}}$ is closed under taking cones of closed degree-zero morphisms;

(b) every object in $\mathcal{A}^{\text{pre-tr}}$ can be obtained from objects in $\mathcal{A}$ by taking successive cones of closed degree-zero morphisms. □

Proof. (a) Given a closed morphism of degree zero

$$f : (\oplus C_i[r_i], q) \longrightarrow (\oplus C'_j[r'_j], q'), \tag{4.21}$$

its cone is the twisted complex $(\oplus C'_j[r'_j] \oplus C_i[r_i + 1], (q', -q + f))$. For example, if $A, B \in \mathcal{A}$ and $f : A \to B$ is a closed morphism of degree zero, then $\mathrm{Cone}(f)$ is the twisted complex $(B \oplus A[1], (0, f)) \in \mathcal{A}^{\text{pre-tr}}$.

(b) Let $C = (\oplus_{i=1}^{n} C_i[r_i], q)$ be a twisted complex. Consider its twisted subcomplex $C' = (\oplus_{i=1}^{n-1} C_i[r_i], q')$, where $q' = q - \oplus_i q_{in}$. Then C is the cone of the closed degree-zero morphism $\oplus_{i=1}^{n-1} q_{in} : (C_n[r_n - 1], 0) \to C'$. ■

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$$\begin{pmatrix} q' & 0 \\ f \circ \sigma^{-1} & -q \end{pmatrix}$$

(Сильно) передтриангульовані dg-категорії

A DG category $\mathcal{A}$ is said to be *pretriangulated* if for every $A \in \mathcal{A}$, $k \in \mathbb{Z}$, the object $A[k] \in \mathcal{A}^{\text{pre-tr}}$ is homotopy-equivalent to an object of $\mathcal{A}$ and for every closed morphism of degree-zero f in $\mathcal{A}$, the object $\mathrm{Cone}(f) \in \mathcal{A}^{\text{pre-tr}}$ is homotopy-equivalent to an object of $\mathcal{A}$. We say that $\mathcal{A}$ is *strongly pretriangulated* if the same is true with “homotopy-equivalent” replaced by “DG isomorphic.” Actually, if $\mathcal{A}$ is pretriangulated (resp., strongly pretriangulated), then every object of $\mathcal{A}^{\text{pre-tr}}$ is homotopy-equivalent (resp., DG isomorphic) to an object of $\mathcal{A}$ [9]. Thus, $\mathcal{A}$ is pretriangulated (resp., strongly pretriangulated) if and only if the embedding $\mathrm{Ho}(\mathcal{A}) \hookrightarrow \mathrm{Ho}(\mathcal{A}^{\text{pre-tr}})$ is an equivalence (resp., the embedding $\mathcal{A} \hookrightarrow \mathcal{A}^{\text{pre-tr}}$ is a DG equivalence).

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Definition 3.3. A dg category $\mathcal{C}$ is *strongly pretriangulated* if $A[n]$ and $\text{Cone}\,(f)$ exist (in $\mathcal{C}$), for every $n \in \mathbb{Z}$, every object A of $\mathcal{C}$ and every morphism f of $Z^0(\mathcal{C})$.

A dg category $\mathcal{C}$ is *pretriangulated* if there exists a quasi-equivalence $\mathcal{C} \to \mathcal{C}'$ with $\mathcal{C}'$ strongly pretriangulated.

dgMod($\mathcal{C}$) – сильно передтриангульована dg-категорія

Let $\phi : \mathrm{M} \to \mathrm{N} \in \mathrm{Z}^0\mathrm{dgMod}(\mathcal{C})$. It is given by the family $\phi(\mathrm{X}) : \mathrm{M}(\mathrm{X}) \to \mathrm{N}(\mathrm{X}) \in \mathsf{dg}$ such that for all $\mathrm{f} \in \mathcal{C}(\mathrm{X},\mathrm{Y})^{\bullet}$

$$\begin{array}{ccc} \mathrm{M}(\mathrm{X}) & \xrightarrow{\mathrm{M}(\mathrm{f})} & \mathrm{M}(\mathrm{Y}) \\ {\scriptstyle \phi(\mathrm{X})}\downarrow & = & \downarrow{\scriptstyle \phi(\mathrm{Y})} \\ \mathrm{N}(\mathrm{X}) & \xrightarrow{\mathrm{N}(\mathrm{f})} & \mathrm{N}(\mathrm{Y}) \end{array}$$

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Define $\mathcal{C}$-module $\mathsf{Cone}\,\phi$ by
$(\mathsf{Cone}\,\phi)(\mathrm{X}) = \mathsf{Cone}(\phi(\mathrm{X})) \equiv \mathsf{Cone}\,\phi(\mathrm{X}) = (\mathrm{M}(\mathrm{X})[1] \oplus \mathrm{N}(\mathrm{X}), \mathrm{d})$
and $\mathcal{C}^{\mathsf{op}}(\mathrm{X}, \mathrm{Y}) \to \underline{\mathsf{dg}}(\mathsf{Cone}\,\phi(\mathrm{X}), \mathsf{Cone}\,\phi(\mathrm{Y}))$, $\mathrm{f} \mapsto \mathrm{M}(\mathrm{f})[1] \oplus \mathrm{N}(\mathrm{f})$.

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Exercise

Verify that this is a chain map.

Ортогональні категорії

<u>6-2 Proposition</u> : Soit N une sous-catégorie triangulée d'une catégorie triangulée A. La catégorie pleine $N^{\perp}$ (resp ${}^{\perp}N$) engendrée par les objets X de A tels que pour tout objet Y de N on ait $\mathrm{Hom}_A(Y,X) = 0$ (resp $\mathrm{Hom}_A(X,Y) = 0$), est une sous-catégorie épaisse de A. La catégorie $N^{\perp}$ est appelée par abus de langage, l'orthogonale à droite de N.

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<u>1-1 Définition</u> : Une sous-catégorie B d'une catégorie triangulée A est dite épaisse si B est une sous-catégorie triangulée pleine de A et si de plus B possède la propriété suivante :

Pour tout morphisme $f : X \longrightarrow Y$, se factorisant par un objet de B et contenu dans un triangle distingué (X,Y,Z,f,g,h) où Z est un objet de B , la source de f et le but de f sont des objets de B .

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(ii) *For any* $W \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}_{\mathscr{C}}(W, \cdot)$ *and* $\mathrm{Hom}_{\mathscr{C}}(\cdot, W)$ *are cohomological functors.*

Proof of Proposition 6-2.

Let $\mathcal{N} \subset \mathcal{A}$ be a full subcategory closed wrt shifts. Let a full subcategory $\mathcal{L} = {}^{\perp}\mathcal{N}$ consist of $\mathrm{X} \in \mathsf{Ob}\,\mathcal{A}$ s/t $\mathcal{A}(\mathrm{X}, \mathcal{N}) = 0$. $\forall \mathrm{N} \in \mathcal{N}$ and for any distinguished triangle $\mathrm{X} \to \mathrm{Y} \to \mathrm{Z} \to$ in $\mathcal{A}$ there is an exact sequence

$$\mathcal{A}(\mathrm{X}, \mathrm{N}[-1]) \to \mathcal{A}(\mathrm{Z}, \mathrm{N}) \to \mathcal{A}(\mathrm{Y}, \mathrm{N}) \to \mathcal{A}(\mathrm{X}, \mathrm{N}) \to \mathcal{A}(\mathrm{Z}, \mathrm{N}[1]).$$

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If two of three vertices of the triangle are in $\mathcal{L}$ so is the third. Thus $\mathcal{L}$ is a triangulated subcategory of $\mathcal{A}$.

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Assume that $Z \in \mathcal{L}$, $N \in \mathcal{N}$ and $f : X \to Y$ factorises through $L \in \mathcal{L}$. Then the exact sequence implies that $\mathcal{A}(f, N) : \mathcal{A}(Y, N) \to \mathcal{A}(X, N)$ is a bijection. On the other hand,

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Hence, $\mathcal{A}(X, N) = 0$ and $\mathcal{A}(Y, N) = 0$. Therefore, $\mathcal{L}$ is épaisse. $\square$

Досконалі модулі

If $\mathcal{C}$ is a dg category, $\mathrm{dgMod}(\mathcal{C})$, $\mathrm{dgAcy}(\mathcal{C})$ and $\mathrm{h\text{-}proj}(\mathcal{C})$ are strongly pretriangulated dg categories. Moreover, the (triangulated) categories $H^0(\mathrm{dgMod}(\mathcal{C}))$, $H^0(\mathrm{dgAcy}(\mathcal{C}))$ and $H^0(\mathrm{h\text{-}proj}(\mathcal{C}))$ have arbitrary coproducts, and there is a semi-orthogonal decomposition

$$H^0(\mathrm{dgMod}(\mathcal{C})) = \langle H^0(\mathrm{dgAcy}(\mathcal{C})), H^0(\mathrm{h\text{-}proj}(\mathcal{C})\rangle. \tag{3.1}$$

This clearly implies that there is an exact equivalence between $H^0(\mathrm{h\text{-}proj}(\mathcal{C}))$ and the Verdier quotient $\mathcal{D}(\mathcal{C}) := H^0(\mathrm{dgMod}(\mathcal{C}))/H^0(\mathrm{dgAcy}(\mathcal{C}))$ (which is by definition the *derived category* of $\mathcal{C}$).

For every dg category $\mathcal{C}$ we will denote by $\mathrm{Pretr}(\mathcal{C})$ (respectively, $\mathrm{Perf}(\mathcal{C})$) the smallest full dg subcategory of $\mathrm{h\text{-}proj}(\mathcal{C})$ containing $\mathsf{Y}^{\mathcal{C}}_{\mathrm{dg}}(\mathcal{C})$ and closed under homotopy equivalences, shifts, cones (respectively, also direct summands in $H^0(\mathrm{h\text{-}proj}(\mathcal{C}))$). It is easy to see that $\mathrm{Pretr}(\mathcal{C})$ and $\mathrm{Perf}(\mathcal{C})$ are strongly pretriangulated and that $\mathcal{C}$ is pretriangulated if and only if $\mathsf{Y}^{\mathcal{C}}_{\mathrm{dg}}\colon \mathcal{C} \to \mathrm{Pretr}(\mathcal{C})$ is a quasi-equivalence. Moreover, $\mathrm{Pretr}(\mathcal{C}) \subseteq \mathrm{Perf}(\mathcal{C})$ and $H^0(\mathrm{Perf}(\mathcal{C}))$ can be identified with the idempotent completion $H^0(\mathrm{Pretr}(\mathcal{C}))^{\mathrm{ic}}$ of $H^0(\mathrm{Pretr}(\mathcal{C}))$. Hence $\mathsf{Y}^{\mathcal{C}}_{\mathrm{dg}}\colon \mathcal{C} \to \mathrm{Perf}(\mathcal{C})$ is a quasi-equivalence if and only if $\mathcal{C}$ is pretriangulated and $H^0(\mathcal{C})$ is idempotent complete.

Remark 3.5. Recall that an additive category $\mathcal{A}$ is *idempotent complete* if every idempotent (namely, a morphism $e\colon A \to A$ in $\mathcal{A}$ such that $e^2 = e$) splits, or, equivalently, has a kernel. Every additive category $\mathcal{A}$ admits a fully faithful and additive embedding $\mathcal{A} \hookrightarrow \mathcal{A}^{\mathrm{ic}}$, where $\mathcal{A}^{\mathrm{ic}}$ is an idempotent complete additive category, with the property that every object of $\mathcal{A}^{\mathrm{ic}}$ is a direct summand of an object from $\mathcal{A}$. The category $\mathcal{A}^{\mathrm{ic}}$ (or, better, the functor $\mathcal{A} \to \mathcal{A}^{\mathrm{ic}}$) is called the *idempotent completion* of $\mathcal{A}$. It can be proved (see [2]) that, if $\mathcal{T}$ is a triangulated category, then $\mathcal{T}^{\mathrm{ic}}$ is triangulated as well (and $\mathcal{T} \hookrightarrow \mathcal{T}^{\mathrm{ic}}$ is exact).

Досконалі модулі

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For every dg category $\mathcal{C}$ we will denote by $\text{Pretr}(\mathcal{C})$ (respectively, $\text{Perf}(\mathcal{C})$) the smallest full dg subcategory of $\text{h-proj}(\mathcal{C})$ containing $\mathsf{Y}^{\mathcal{C}}_{\text{dg}}(\mathcal{C})$ and closed under homotopy equivalences, shifts, cones (respectively, also direct summands in $H^0(\text{h-proj}(\mathcal{C}))$). It is easy to see that $\text{Pretr}(\mathcal{C})$ and $\text{Perf}(\mathcal{C})$ are strongly pretriangulated and that $\mathcal{C}$ is pretriangulated if and only if $\mathsf{Y}^{\mathcal{C}}_{\text{dg}}\colon \mathcal{C} \to \text{Pretr}(\mathcal{C})$ is a quasi-equivalence. Moreover, $\text{Pretr}(\mathcal{C}) \subseteq \text{Perf}(\mathcal{C})$ and $H^0(\text{Perf}(\mathcal{C}))$ can be identified with the idempotent completion $H^0(\text{Pretr}(\mathcal{C}))^{\text{ic}}$ of $H^0(\text{Pretr}(\mathcal{C}))$. Hence $\mathsf{Y}^{\mathcal{C}}_{\text{dg}}\colon \mathcal{C} \to \text{Perf}(\mathcal{C})$ is a quasi-equivalence if and only if $\mathcal{C}$ is pretriangulated and $H^0(\mathcal{C})$ is idempotent complete.

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$$\forall N \in \text{dgMod}(\mathcal{C})\ \exists M \in \text{h-proj}(\mathcal{C})\ \exists M \xrightarrow{f} N \to \mathsf{Cone}\, f, \in \text{dgAcy}(\mathcal{C})$$

Досконалі комплекси

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An object X in a category $\mathcal{C}$ which admits all filtered colimits is called compact if the functor

$$\mathcal{C}(X, \cdot) : \mathcal{C} \to \mathrm{Sets}, Y \mapsto \mathcal{C}(X, Y)$$

commutes with filtered colimits, i.e., if the natural map

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is a bijection for any filtered system of objects Y_i in $\mathcal{C}$.

Since elements in the filtered colimit at the left are represented by maps $\mathrm{X} \to \mathrm{Y}_\mathrm{i}$, for some i, the surjectivity of the above map amounts to requiring that a map $\mathrm{X} \to \mathsf{colim}_\mathrm{i}\, \mathrm{Y}_\mathrm{i}$ factors over some Y_i.

Добре породжені триангульовані категорії

Let $\mathcal{T}$ be a triangulated category with small coproducts. For a cardinal α, an object S of $\mathcal{T}$ is α-small if every map $\mathrm{S} \to \coprod_{\mathrm{i}\in\mathrm{I}} \mathrm{X_i}$ in $\mathcal{T}$ (where I is a small set) factors through $\coprod_{\mathrm{i}\in\mathrm{J}} \mathrm{X_i}$, for some $\mathrm{J} \subseteq \mathrm{I}$ with $|\mathrm{J}| < \alpha$. A cardinal α is called regular if it is not the sum of fewer than α cardinals, all of them smaller than α.

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Definition

The category $\mathcal{T}$ is well generated if there exists a small set $\mathcal{S}$ of objects in $\mathcal{T}$ satisfying the following properties:

(G1) An object $\mathrm{X} \in \mathcal{T}$ is isomorphic to 0, if and only if $\mathcal{T}(\mathrm{S}, \mathrm{X}[\mathrm{j}]) = 0$, for all $\mathrm{S} \in \mathcal{S}$ and all $\mathrm{j} \in \mathbb{Z}$;

(G2) For every small set of maps $\{\mathrm{X}_\mathrm{i} \to \mathrm{Y}_\mathrm{i}\}_{\mathrm{i}\in\mathrm{I}}$ in $\mathcal{T}$, the induced map $\mathcal{T}(\mathrm{S}, \coprod_\mathrm{i} \mathrm{X}_\mathrm{i}) \to \mathcal{T}(\mathrm{S}, \coprod_\mathrm{i} \mathrm{Y}_\mathrm{i})$ is surjective for all $\mathrm{S} \in \mathcal{S}$, if $\mathcal{T}(\mathrm{S}, \mathrm{X}_\mathrm{i}) \to \mathcal{T}(\mathrm{S}, \mathrm{Y}_\mathrm{i})$ is surjective, for all $\mathrm{i} \in \mathrm{I}$ and all $\mathrm{S} \in \mathcal{S}$;

(G3) There exists a regular cardinal α such that every object of $\mathcal{S}$ is α-small.

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