6. Напіввільні модулі над dg-категорією
Навколо похідних категорій

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B.1. Definition. A DG R-module F over a DG ring R is *free* if it is isomorphic to a direct sum of DG modules of the form $R[n]$, $n \in \mathbb{Z}$. A DG R-module F is *semi-free* if the following equivalent conditions hold:

(1) F can be represented as the union of an increasing sequence of DG submodules F_i, $i = 0, 1, \ldots$, so that $F_0 = 0$ and each quotient F_i / F_{i-1} is free;
A complex of k-modules is semi-free if it is semi-free as a DG k-module.

B.2. Remarks.

(i) A bounded above complex of free k-modules is semi-free.

Theorem

Припустимо, що S - це множина, категорія $\mathcal{C}$ є повною і коповною і $F : \mathsf{dg}^S \rightleftarrows \mathcal{C} : U$ є спряженням. Припустимо, що U зберігає фільтруючі кограниці. Для будь-якого $x \in S$ розглянемо об'єкт $\mathbb{K}_x$ з dg^S, $\mathbb{K}_x(x) = \mathsf{Cone}(\mathsf{id}_{\Bbbk})$, $\mathbb{K}_x(y) = 0$ для $y \neq x$. Припустимо, що ланцюгове відображення $U(\mathsf{in}_2) : UA \to U(F(\mathbb{K}_x[p]) \sqcup A)$ - квазіізоморфізм для всіх об'єктів A з $\mathcal{C}$ і всіх $x \in S$, $p \in \mathbb{Z}$. Оснастимо $\mathcal{C}$ класами слабких еквівалентів (відповідно фібрацій), що складаються з морфізмів f з $\mathcal{C}$ таких що Uf - квазіізоморфізм (відповідно епіморфізм). Тоді категорія $\mathcal{C}$ - модельна категорія.

Застосування теореми Хініча

S = one-element set,
$\mathcal{C} = \mathsf{mod}\text{-}\mathrm{R}$ – right dg-modules over dg $\Bbbk$-algebra R,
$\mathcal{C}$ is additive (even abelian) $\Rightarrow$ for finite I $\coprod_{\mathrm{i}\in\mathrm{I}} = \oplus_{\mathrm{i}\in\mathrm{I}}$.
$\mathcal{C}$ is complete and cocomplete.
Additive functors $\mathrm{F} : \mathsf{dg} \rightleftarrows \mathcal{C} : \mathrm{U}$ are:
free R-module $\mathrm{F} : \mathrm{M} \mapsto \mathrm{M} \otimes_{\Bbbk} \mathrm{R}$
underlying complex $(\mathrm{P}^\bullet \in \mathsf{dg}_{\mathbb{Z}}, \Bbbk \to \mathrm{R} \to \underline{\mathsf{dg}}_{\mathbb{Z}}(\mathrm{P},\mathrm{P})) \longleftarrow\!\!\dashv \mathrm{P} : \mathrm{U}$,
U зберігає фільтруючі кограниці.
$\mathrm{F} : \mathsf{dg}^{\mathrm{S}} \rightleftarrows \mathcal{C} : \mathrm{U}$ є спряженням: $\mathcal{C}(\mathrm{FM},\mathrm{P}) \cong \mathsf{dg}(\mathrm{M},\mathrm{UP})$ природ.
$\mathrm{F}(\Bbbk[\mathrm{n}]) = \mathrm{R}[\mathrm{n}]$. We have
$\mathrm{U}(\mathsf{in}_2) = \mathsf{in}_2 : \mathrm{UP} \to \mathrm{U}(\mathrm{F}(\mathbb{K}_{\mathrm{x}}[\mathrm{p}]) \sqcup \mathrm{P}) = \mathrm{UF}(\mathbb{K}[\mathrm{p}]) \oplus \mathrm{UP}$.
The summand $\mathrm{U}(\mathrm{F}(\mathbb{K}[\mathrm{p}]) = 0 \to \underset{-\mathrm{p}}{\mathrm{R}} \xrightarrow{1} \underset{-\mathrm{p}+1}{\mathrm{R}} \to 0$ is contractible
$\Rightarrow \mathrm{U}(\mathsf{in}_2)$ is a quasi-isomorphism.
Conditions of the theorem are satisfied.

Нехай $\mathrm{M} \in \mathsf{Ob}\,\mathsf{dg}$, $\mathrm{A} \in \mathsf{Ob}\,\mathcal{C}$, $\alpha : \mathrm{M} \to \mathrm{A}^{\#} \in \mathsf{dg}$. Позначимо через $\mathrm{C} = \mathsf{Cone}\,\alpha = (\mathrm{M}[1] \oplus \mathrm{UA}, \mathrm{d}_{\mathsf{Cone}}) \in \mathsf{Ob}\,\mathsf{dg}$ конус. Позначимо через $\bar{\imath} = \mathsf{in}_2 : \mathrm{UA} \to \mathrm{C}$ очевидне вкладення. Означимо об'єкт $\mathrm{A}\langle \mathrm{M}, \alpha \rangle \in \mathsf{Ob}\,\mathcal{C}$ як виштовхування

$$\begin{array}{ccc}
\mathrm{A} \otimes_{\Bbbk} \mathrm{R} & \xrightarrow{\text{action}} & \mathrm{A} \\
{\scriptstyle \bar{\imath}\otimes 1}\downarrow & & \downarrow{\scriptstyle \bar{\jmath}} \\
\mathrm{C} \otimes_{\Bbbk} \mathrm{R} & \xrightarrow{\mathrm{g}} & \mathrm{A}\langle \mathrm{M}, \alpha \rangle
\end{array}$$

Отже, $\mathrm{A}\langle \mathrm{M}, \alpha \rangle = ((\mathrm{M}[1] \otimes_{\Bbbk} \mathrm{R}) \oplus \mathrm{UA}, \mathrm{d})$, де

$$\mathrm{d} = \begin{pmatrix} \mathrm{d}_{\mathrm{M}[1]\otimes_{\Bbbk}\mathrm{R}} & \mathrm{M}[1] \otimes_{\Bbbk} \mathrm{R} \xrightarrow{\sigma^{-1}\otimes 1} \mathrm{M} \otimes_{\Bbbk} \mathrm{R} \xrightarrow{\alpha\otimes 1} \mathrm{A} \otimes_{\Bbbk} \mathrm{R} \xrightarrow{\text{action}} \mathrm{A} \\ 0 & \mathrm{d}_{\mathrm{A}} \end{pmatrix}.$$

Отже, $0 \to \mathrm{A} \to \mathrm{A}\langle \mathrm{M}, \alpha \rangle \to \mathrm{M}[1] \otimes_{\Bbbk} \mathrm{R} \to 0$ точна в $\mathsf{dg}\,\mathsf{mod}\text{-}\,\mathrm{R}$ і напіврозщеплювана = розщеплювана в $\mathsf{gr}\,\mathsf{mod}\text{-}\,\mathrm{R}$.
Нам потрібен M з $\mathrm{d}_{\mathrm{M}} = 0$. Тоді $\mathrm{d}_{\mathrm{M}[1]\otimes_{\Bbbk}\mathrm{R}} = 1_{\mathrm{M}[1]} \otimes \mathrm{d}_{\mathrm{R}}$.
$\Rightarrow \mathcal{C}$ has model structure in which semi-free modules are cofibrant.

dg-категорії зі скінченною множиною об'єктів

A dg $\Bbbk$-algebra R is associated with a dg-category $\mathcal{A}$ with finite $\mathsf{Ob}\,\mathcal{A}$:

$$R = \bigoplus_{i,j\in\mathsf{Ob}\,\mathcal{A}} \mathcal{A}(i,j) \qquad (\text{with } \mathcal{A}(i,j)\cdot\mathcal{A}(k,l) = 0 \text{ if } j\neq k).$$

This algebra has a family of pairwise orthogonal idempotents $(e_i = \mathsf{id}_i)_{i\in\mathsf{Ob}\,\mathcal{A}}$ such that $\sum_{i\in\mathsf{Ob}\,\mathcal{A}} e_i = 1$ and $e_i d = 0$.
$\Rightarrow$ Any R-module P splits into dg $\Bbbk$-subcomplexes $P_i = Pe_i$ with the action $P_i \otimes_{\Bbbk} \mathcal{A}(i,j) \to P_j$.
The latter is equivalent to a chain map $\mathcal{A}(i,j) \to \underline{\mathsf{dg}}^r(P_i, P_j)$.
Together with the function $i \mapsto Pe_i$ these define a dg-functor $P^* : \mathcal{A} \to \mathsf{dg}_{\Bbbk}$.
The correspondence $P \mapsto P^*$ extends to an equivalence $\mathsf{mod}\text{-}R = \mathsf{dg}\,\mathsf{mod}\text{-}R \to \mathsf{dg}\mathcal{C}\mathsf{at}(\mathcal{A}, \mathsf{dg}) = \mathrm{dgMod}(\mathcal{A})$.
For instance, take the R-module $P = jR = e_jR$. Then $P^*(i) = e_jRe_i = \mathcal{A}(j,i)$. $\Rightarrow$ $(e_jR)^* = \tilde{h}_j$, where $\tilde{h}_j(i) = \mathcal{A}(j,i)$.

Модулі над dg-категорією

C.1. Let $\mathcal{A}$ be a DG category. A *left DG $\mathcal{A}$-module* is a DG functor from $\mathcal{A}$ to the DG category of complexes of k-modules. Sometimes left DG $\mathcal{A}$-modules will be called simply DG $\mathcal{A}$-modules. If $\mathcal{A}$ has a single object U with $\operatorname{End}_{\mathcal{A}} U = R$ then a DG $\mathcal{A}$-module is the same as a DG R-module. A *right DG $\mathcal{A}$-module* is a left DG module over the dual DG category $\mathcal{A}^{\circ}$. The DG category of DG $\mathcal{A}$-modules is denoted by $\mathcal{A}$-DGmod. In particular, k-DGmod is the DG category of complexes of k-modules.

C.2. Let $\mathcal{A}$ be a DG category. Then the complex

$$\operatorname{Alg}_{\mathcal{A}} := \bigoplus_{X,Y\in \operatorname{Ob}\mathcal{A}} \operatorname{Hom}(X,Y)$$

has a natural DG algebra structure (interpret elements of $\operatorname{Alg}_{\mathcal{A}}$ as matrices (f_{XY}), $f_{XY} \in \operatorname{Hom}(Y,X)$, whose rows and columns are labeled by $\operatorname{Ob}\mathcal{A}$). The DG algebra $\operatorname{Alg}_{\mathcal{A}}$ has the following property: every finite subset of $\operatorname{Alg}_{\mathcal{A}}$ is contained in $e \operatorname{Alg}_{\mathcal{A}} e$ for some idempotent $e \in \operatorname{Alg}_{\mathcal{A}}$ such that $de = 0$ and $\deg e = 0$. We say that a module M over $\operatorname{Alg}_{\mathcal{A}}$ is *quasi-unital* if every element of M belongs to eM for some idempotent $e \in \operatorname{Alg}_{\mathcal{A}}$ (which may be assumed closed of degree 0 without loss of generality). If Φ is a DG $\mathcal{A}$-module then $M_{\Phi} := \bigoplus_{X\in\operatorname{Ob}\mathcal{A}} \Phi(X)$ is a DG module over $\operatorname{Alg}_{\mathcal{A}}$ (to define multiplication write elements of $\operatorname{Alg}_{\mathcal{A}}$ as matrices and elements of M_{Φ} as columns). Thus, we get a DG equivalence between the DG category of DG $\mathcal{A}$-modules and that of quasi-unital DG modules over $\operatorname{Alg}_{\mathcal{A}}$.

Представлювані модулі

C.3. Let $F : \mathcal{A} \to k$-DGmod be a left DG $\mathcal{A}$-module and $G : \mathcal{A} \to k$-DGmod a right DG $\mathcal{A}$-module. A *DG pairing* $G \times F \to C$, $C \in k$-DGmod, is a DG morphism from the DG bifunctor $(X, Y) \mapsto \mathrm{Hom}(X, Y)$ to the DG bifunctor $(X, Y) \mapsto \mathrm{Hom}(G(Y) \otimes F(X), C)$. It can be equivalently defined as a DG morphism $F \to \mathrm{Hom}(G, C)$ or as a DG morphism $G \to \mathrm{Hom}(F, C)$, where $\mathrm{Hom}(G, C)$ is the DG functor $X \mapsto \mathrm{Hom}(G(X), C)$, $X \in \mathcal{A}$. There is a universal DG pairing $G \times F \to C_0$. We say that C_0 is the *tensor product* of G and F, and we write $C_0 = G \otimes_{\mathcal{A}} F$. Explicitly, $G \otimes_{\mathcal{A}} F$ is the quotient of $\bigoplus_{X \in \mathcal{A}} G(X) \otimes F(X)$ by the following relations: for every morphism $f : X \to Y$ in $\mathcal{A}$ and every $u \in G(Y)$, $v \in F(X)$ one should identify $f^*(u) \otimes v$ and $u \otimes f_*(v)$. In terms of [39, §IX.6], $G \otimes_{\mathcal{A}} F = \int^X G(X) \otimes F(X)$, i.e., $G \otimes_{\mathcal{A}} F$ is the coend of the functor $\mathcal{A}^\circ \times \mathcal{A} \to k$-DGmod defined by $(Y, X) \mapsto G(Y) \otimes F(X)$. In terms of C.2, a DG pairing $G \times F \to C$ is the same as a DG pairing $M_G \times M_F \to C$, so $G \otimes_{\mathcal{A}} F = M_G \otimes_{\mathrm{Alg}_{\mathcal{A}}} M_F$.

C.4. Example. For every $Y \in \mathcal{A}$ one has the right DG $\mathcal{A}$-module h_Y and the left DG $\mathcal{A}$-module $\tilde{h}_Y$ defined by $h_Y(Z) := \mathrm{Hom}(Z, Y)$, $\tilde{h}_Y(Z) := \mathrm{Hom}(Y, Z)$, $Z \in \mathcal{A}$. One has the canonical isomorphisms

$$G \otimes_{\mathcal{A}} \tilde{h}_Y = G(Y), \tag{C.1}$$

$$h_Y \otimes_{\mathcal{A}} F = F(Y) \tag{C.2}$$

induced by the maps $G(Z) \otimes \mathrm{Hom}(Y, Z) \to G(Y)$, $\mathrm{Hom}(Z, Y) \otimes F(Z) \to F(Y)$, $Z \in \mathcal{A}$.

Ациклічні модулі

C.5. Given DG categories $\mathcal{A}, \mathcal{B}, \overline{\mathcal{B}}$, a DG $\mathcal{A} \otimes \mathcal{B}$-module F, and a DG $(\mathcal{A}^\circ \otimes \overline{\mathcal{B}})$-module G, one defines the DG $\overline{\mathcal{B}} \otimes \mathcal{B}$-module $G \otimes_{\mathcal{A}} F$ as follows. We consider F as a DG functor from $\mathcal{B}$ to the DG category of DG $\mathcal{A}$-modules, so $F(X)$ is a DG $\mathcal{A}$-module for every $X \in \mathcal{B}$. Quite similarly, $G(Y)$ is a DG $(\mathcal{A})^\circ$-module for every $Y \in \overline{\mathcal{B}}$. Now $G \otimes_{\mathcal{A}} F$ is the DG functor $Y \otimes X \mapsto G(Y) \otimes_{\mathcal{A}} F(X)$, $X \in \mathcal{B}$, $Y \in \overline{\mathcal{B}}$.

C.6. Denote by $Hom_{\mathcal{A}}$ the DG $\mathcal{A} \otimes \mathcal{A}^\circ$-module $(X, Y) \mapsto \mathrm{Hom}(Y, X)$, $X, Y \in \mathcal{A}$. E.g., if $\mathcal{A}$ has a single object and R is its DG algebra of endomorphisms then $Hom_{\mathcal{A}}$ is the DG R-bimodule R. For any DG category $\mathcal{A}$ the isomorphisms (C.1) and (C.2) induce canonical isomorphisms

$$Hom_{\mathcal{A}} \otimes_{\mathcal{A}} F = F, \qquad G \otimes_{\mathcal{A}} Hom_{\mathcal{A}} = G \tag{C.3}$$

for every left DG $\mathcal{A}$-module F and right DG $\mathcal{A}$-module G

C.7. A left or right DG $\mathcal{A}$-module $F : \mathcal{A} \to k$-DGmod is said to be *acyclic* if the complex $F(X)$ is acyclic for every $X \in \mathcal{A}$. A left DG $\mathcal{A}$-module F is said to be *homotopically flat* if $G \otimes_{\mathcal{A}} F$ is acyclic for every acyclic right DG $\mathcal{A}$-module G. A right DG $\mathcal{A}$-module is said to be homotopically flat if it is homotopically flat as a left DG $\mathcal{A}^\circ$-module. It follows from (C.1) and (C.2) that h_Y and $\tilde{h}_Y$ are homotopically flat.

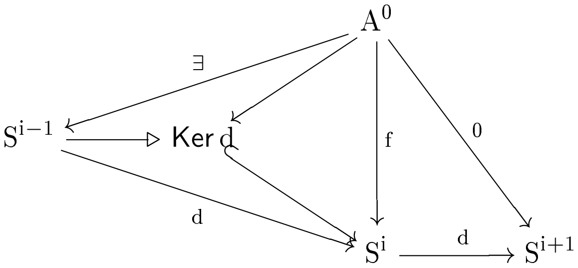
DEFINITION. A complex $A^{\cdot}$ (in $\mathfrak{C}$ or $\mathfrak{K}$) is *K-projective* (resp. *K-injective*) if for every acyclic complex $S^{\cdot} \in \mathfrak{C}$, the complex $\mathrm{Hom}^{\cdot}(A^{\cdot}, S^{\cdot})$ (resp. $\mathrm{Hom}^{\cdot}(S^{\cdot}, A^{\cdot})$) is acyclic.

1.2. PROPOSITION. *Let $A^{\cdot} \in \mathfrak{C}$ be such that $A^i = 0$ for $i \neq 0$. Then $A^{\cdot}$ is K-projective (resp. K-injective) if and only if A^0 is a projective (resp. injective) object of $\mathfrak{A}$.*

Доведення. Let A^0 be a projective module, Then $0 \to A^0 \to 0$ is homotopy projective: Let $\ldots \to S^{i-1} \to S^i \to S^{i+1} \to \ldots$ be acyclic. We have to prove that

$$\ldots \to \Bbbk\text{-mod}(A^0, S^{i-1}) \to \Bbbk\text{-mod}(A^0, S^i) \to \Bbbk\text{-mod}(A^0, S^{i+1}) \to \ldots$$

is acyclic. This follows from projectivity of A^0 and diagram



□

Напіввільні модулі гомотопічно проективні

Over the dg ring R

As noticed in [1,19], a semi-free DG module F is *homotopically projective*, which means that for every acyclic DG module N every morphism $f: F \to N$ is homotopic to 0 (we prefer to use the name "homotopically projective" instead of Spaltenstein's name "K-projective"). Indeed, if $\{F_i\}$ is a filtration on F satisfying the condition from B.1, then every homotopy between $f|_{F_{i-1}}$ and 0 can be extended to a homotopy between $f|_{F_i}$ and 0. This also follows from Lemma 4.4 applied to the triangulated

Proposition

A semi-free DG R-module F is homotopically projective.

Доведення. Indeed, if $\{F_i\}$ is a filtration on F, then every homotopy between $f|_{F_{i-1}}$ and 0 can be extended to a homotopy between $f|_{F_i}$ and 0. The exact sequence $0 \to F_{i-1} \to F_i \xrightarrow{\psi} P \to 0$ in $\mathsf{dg\,mod\text{-}}R$ (where $P = \oplus_{j\in J} R[n_j] = \{p_j \mid j \in J\}R$, $\deg p_j = -n_j$) is semisplit, that is, split by a map $k: P \to F_i \in \mathsf{gr\,mod\text{-}}R$ commuting with the action of R, but not with the differential. We are given $f: F_i \to N \in \mathsf{dg\,mod\text{-}}R$ with acyclic N and homotopy $h': F_{i-1} \to N \in \mathsf{gr\,mod\text{-}}R$ such that $f|_{F_{i-1}} = d \cdot h' + h' \cdot d$.

Напіввільні модулі над dg-категорією

We look for degree -1 map $\mathrm{h} : \mathrm{F_i} \to \mathrm{N} \in \mathsf{gr\,mod}\text{-}\mathrm{R}$ such that $\mathrm{h}|_{\mathrm{F_{i-1}}} = \mathrm{h}'$ and $\mathrm{f} = \mathrm{d} \cdot \mathrm{h} + \mathrm{h} \cdot \mathrm{d}$. It suffices to specify h on $\mathrm{p_j k}$. Notice that $\mathrm{p_j k d}\psi = \mathrm{p_j k}\psi\mathrm{d} = \mathrm{p_j d} = 0$ implies $\mathrm{p_j k d} \in \mathrm{F_{i-1}}$. One can find $\mathrm{p_j k h} \in \mathrm{N}$ which solves $\mathrm{p_j k(f - d \cdot h') = (p_j k h)d}$ since

$$\mathrm{p_j k(f{-}d{\cdot}h')d = p_j kd(f{-}h'{\cdot}d) = p_j kd(f|_{F_{i-1}}{-}h'{\cdot}d) = p_j kddh' = 0.}$$

The homotopy $\mathrm{h} : \mathrm{F} \to \mathrm{N} \in \mathsf{gr\,mod}\text{-}\mathrm{R}$ is constructed by induction. □

C.8. Let $\mathcal{A}$ be a DG category. A DG $\mathcal{A}$-module is said to be *free* if it is isomorphic to a direct sum of complexes of the form $\tilde{h}_X[n]$, $X \in \mathcal{A}$, $n \in \mathbb{Z}$. The notion of semi-free DG $\mathcal{A}$-module is quite similar to that of semi-free module over a DG algebra (see Definition B.1): an $\mathcal{A}$-module Φ is said to be *semi-free* if it can be represented as the union of an increasing sequence of DG submodules Φ_i, $i = 0, 1, \ldots$, so that $\Phi_0 = 0$ and each quotient Φ_i/Φ_{i-1} is free. Clearly, a semi-free DG $\mathcal{A}$-module is homotopically flat. For every DG $\mathcal{A}$-module Φ_i there is a quasi-isomorphism $F \to \Phi$ such that F is a semi-free DG $\mathcal{A}$-module; this is proved just as in the case that $\mathcal{A}$ has a single object (see Lemma B.3). Just as in Remarks B.2, one shows that a semi-free DG $\mathcal{A}$-module is homotopically projective (i.e., the complex $\mathrm{Hom}(F, N)$ is acyclic for every acyclic DG $\mathcal{A}$-module N) and that the functor from the homotopy category of semi-free DG $\mathcal{A}$-modules to the derived category $D(\mathcal{A}^\circ)$ of $\mathcal{A}$-modules is an equivalence.

Напіввільні модулі гомотопічно пласкі

Proposition

A semi-free $\mathcal{A}$-module F is homotopy flat.

Доведення. Let $\{F_i\}$ be a filtration on F. Since the exact sequence $0 \to F_{i-1} \to F_i \to P \to 0$ ($P = \oplus_{j \in J} \tilde{h}_{Y_j}[n_j]$) is semisplit, the sequence

$$0 \to G \otimes_{\mathcal{A}} F_{i-1} \to G \otimes_{\mathcal{A}} F_i \to G \otimes_{\mathcal{A}} P \to 0$$

is semisplit as well.
$\Rightarrow$ It is an exact sequence of complexes of $\Bbbk$-modules.
$\Rightarrow$ From the associated long sequence of cohomology groups one concludes:
($G \otimes_{\mathcal{A}} F_{i-1}$ and $G \otimes_{\mathcal{A}} P$ are acyclic $\Rightarrow$ $G \otimes_{\mathcal{A}} F_i$ is acyclic).
$\Rightarrow$ $G \otimes_{\mathcal{A}} F$ is acyclic by induction. □

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