4. Передтриангульовані dg-категорії. Навколо похідних категорій

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Монади

Monads are given by endofunctors with a monoid structure on them. We need to include data: a natural map $\eta_X : X \to TX$ – the unit, and a map $\mu_X : TTX \to TX$ – the multiplication (associative and unital).

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Example

$\mathscr{C} = \Bbbk\text{-mod}$, Z – $\Bbbk$-algebra. Then $- \otimes Z : M \mapsto M \otimes Z$ – monad in $\mathscr{C}$. The multiplication and the unit in monad come from the multiplication and the unit in algebra Z.

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Example

The functor $- \boxtimes \mathscr{Z} = -^{[]} : \mathscr{A} \mapsto \mathscr{A} \boxtimes \mathscr{Z} = \mathscr{A}^{[]}$ – monad in dg$\mathscr{C}$at. The multiplication and the unit in monad come from the multiplication and the unit in algebra $\mathscr{Z}$.

We shall construct another example of a monad in dg$\mathscr{C}$at.

Категорія $\mathscr{A}^{\text{pre-tr}}$

2.4. To a DG category $\mathcal{A}$ Bondal and Kapranov associate a triangulated category $\mathcal{A}^{\text{tr}}$ (or $\text{Tr}^{+}(\mathcal{A})$ in the notation of [4]). It is defined as the homotopy category of a certain DG category $\mathcal{A}^{\text{pre-tr}}$. The idea of the definition of $\mathcal{A}^{\text{pre-tr}}$ is to formally add to $\mathcal{A}$ all cones, cones of morphisms between cones, etc. Here is the precise definition from [4]. The objects of $\mathcal{A}^{\text{pre-tr}}$ are "one-sided twisted complexes," i.e., formal expressions $(\bigoplus_{i=1}^{n} C_i[r_i], q)$, where $C_i \in \mathcal{A}$, $r_i \in \mathbb{Z}$, $n \geqslant 0$, $q = (q_{ij})$, $q_{ij} \in \text{Hom}(C_j, C_i)[r_i - r_j]$ is homogeneous of degree 1, $q_{ij} = 0$ for $i \geqslant j$, $dq + q^2 = 0$. If $C, C' \in \text{Ob}\,\mathcal{A}^{\text{pre-tr}}$, $C = (\bigoplus_{j=1}^{n} C_j[r_j], q)$, $C' = (\bigoplus_{i=1}^{m} C'_i[r'_i], q')$ then the $\mathbb{Z}$-graded k-module $\text{Hom}(C, C')$ is the space of matrices $f = (f_{ij})$, $f_{ij} \in \text{Hom}(C_j, C'_i)[r'_i - r_j]$, and the composition map $\text{Hom}(C, C') \otimes \text{Hom}(C', C'') \to \text{Hom}(C, C'')$ is matrix multiplication. The differential $\text{d}\colon \text{Hom}(C, C') \to \text{Hom}(C, C')$ is defined by $\text{d} f := \text{d}_{\text{naive}} f + q' f - (-1)^l f q$ if $\deg f_{ij} = l$, where $\text{d}_{\text{naive}} f := (\text{d} f_{ij})$.

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Suppose $\mathscr{C}$ is a differential graded category. Then there is a differential graded category $\mathscr{C}^{\mathsf{pre\text{-}tr}}$. Explicit description of that is given below.

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An object X of $\mathscr{C}^{\mathsf{pre\text{-}tr}}$ (twisted complex) consists of the following data: a finite totally ordered set I, a function that assigns to every $i \in I$ an object X_i of $\mathscr{C}$ and an integer n_i, and a family of elements $q_{ij} \in \mathscr{C}(X_i, X_j)[n_j - n_i]$, $i,j \in I$, $i < j$ of degree 1, which satisfy Maurer-Cartan equation: for every $i,j \in I$, $i < j$

$$(-)^{n_j - n_i} q_{ij} s^{n_i - n_j} m_1 s^{n_j - n_i} \\ - \sum_{i<k<j} (q_{ik} \otimes q_{kj})(s^{n_k - n_i} \otimes s^{n_j - n_k})^{-1} \mu s^{n_j - n_i} = 0,$$

where $m_1 = d$ and $\mu = m_2$ are respectively the differential and the multiplication in the original category $\mathscr{C}$. Putting $q_{ij} = 0$ for $i \geq j$ we may extend this equation to all pairs $i,j \in I$. We write briefly $X = (\bigoplus_{i \in I} X_i[n_i], q)$.

Морфізми скручених комплексів

When the strictly upper-triangular matrix q has to be specified explicitly we use the following compact notation:

$$X = \begin{pmatrix} X_1[n_1] & q_{12} & q_{13} & \dots \\ & X_2[n_2] & q_{23} & \dots \\ & & X_3[n_3] & \dots \\ & & & \ddots \end{pmatrix}.$$

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Let $X = (\bigoplus_{i\in I} X_i[n_i], q)$, $Y = (\bigoplus_{j\in J} Y_j[m_j], r)$ be objects of $\mathscr{C}^{\mathsf{pre\text{-}tr}}$. The graded $\Bbbk$-module of morphisms between X and Y is defined as

$$\mathscr{C}^{\mathsf{pre\text{-}tr}}(X,Y) = \prod_{i\in I, j\in J} \mathscr{C}(X_i, Y_j)[m_j - n_i].$$

An element f of $\mathscr{C}^{\mathsf{pre\text{-}tr}}(X,Y)$ is thought as a matrix with entries $f_{ij} \in \mathscr{C}(X_i, Y_j)[m_j - n_i]$, $i \in I$, $j \in J$.

Композиція

The composition map is matrix multiplication. More precisely, let $Z = (\bigoplus_{k\in K} Z_k[\ell_k], p)$ be another object of $\mathscr{C}^{\mathsf{pre\text{-}tr}}$, and let g be a morphism from Y to Z. Then $fg \overset{\mathrm{def}}{=} (f \otimes g)m_2^{\mathsf{pre\text{-}tr}}$ has the entries for $i \in I, k \in K$

$$\big[(f \otimes g)m_2^{\mathsf{pre\text{-}tr}}\big]_{ik} = \sum_{j\in J} (f_{ij} \otimes g_{jk})(s^{m_j - n_i} \otimes s^{\ell_k - m_j})^{-1} \mu s^{\ell_k - n_i}.$$

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Compute $(s^a\otimes s^b)^{-1}$.

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The differential $m_1^{\mathsf{pre\text{-}tr}} : \mathscr{C}^{\mathsf{pre\text{-}tr}}(X,Y) \to \mathscr{C}^{\mathsf{pre\text{-}tr}}(X,Y)$ is given by

$$\begin{aligned}\big[f m_1^{\mathsf{pre\text{-}tr}}\big]_{ij} =& (-)^{m_j-n_i} f_{ij} s^{n_i-m_j} m_1 s^{m_j-n_i} \\ &- \sum_{t\in J}(f_{it}\otimes r_{tj})(s^{m_t-n_i}\otimes s^{m_j-m_t})^{-1}\mu s^{m_j-n_i} \\ &+ (-)^f \sum_{u\in I}(q_{iu}\otimes f_{uj})(s^{n_u-n_i}\otimes s^{m_j-n_u})^{-1}\mu s^{m_j-n_i}\end{aligned}$$

for every $i \in I$, $j \in J$.

Let us denote by $\mathrm{d} = \mathrm{m}_1^{[]}$ the naive differential in $\mathscr{C}^{\mathsf{pre\text{-}tr}}(\mathrm{X},\mathrm{Y})$ defined by

$$[\mathrm{fd}]_{\mathrm{ij}} = \mathrm{f}_{\mathrm{ij}}\mathrm{m}_1^{[]} = (-)^{\mathrm{m_j}-\mathrm{n_i}}\mathrm{f}_{\mathrm{ij}}\mathrm{s}^{\mathrm{n_i}-\mathrm{m_j}}\mathrm{m}_1\mathrm{s}^{\mathrm{m_j}-\mathrm{n_i}}, \quad \mathrm{i}\in\mathrm{I}, \mathrm{j}\in\mathrm{J}.$$

Since the composition $\mathrm{m}_2^{\mathsf{pre\text{-}tr}}$ in $\mathscr{C}^{\mathsf{pre\text{-}tr}}$ consists of matrix composition combined with $\mathrm{m}_2^{[]}$, the differential d is a derivation of it: $\mathrm{m}_2^{\mathsf{pre\text{-}tr}}\mathrm{d} = (1\otimes\mathrm{d}+\mathrm{d}\otimes 1)\mathrm{m}_2^{\mathsf{pre\text{-}tr}}$. Denoting the composition of $\mathrm{f}\in\mathscr{C}^{\mathsf{pre\text{-}tr}}(\mathrm{X},\mathrm{Y})$ and $\mathrm{g}\in\mathscr{C}^{\mathsf{pre\text{-}tr}}(\mathrm{Y},\mathrm{Z})$ simply fg we may write the following expressions for the Maurer-Cartan equation and for the differential in $\mathscr{C}^{\mathsf{pre\text{-}tr}}((\mathrm{X},\mathrm{q}),(\mathrm{Y},\mathrm{r}))$:

$$\mathrm{qd} = \mathrm{q}^2,$$
$$\mathrm{fm}_1^{\mathsf{pre\text{-}tr}} = \mathrm{fd} - \mathrm{fr} + (-)^{\mathrm{f}}\mathrm{qf}.$$

Exercise: $(\mathrm{m}_1^{\mathsf{pre\text{-}tr}})^2 = 0$.

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If $\mathrm{X},\mathrm{Y} \in \mathsf{Ob}\,\mathscr{C}$ and $\mathrm{f} : \mathrm{X} \to \mathrm{Y}$ is a closed morphism of degree 0 one defines $\mathsf{Cone}(\mathrm{f})$ to be the object $\left(\mathrm{X}[1] \oplus \mathrm{Y}, \left(\begin{smallmatrix} 0 & \mathrm{f} \\ 0 & 0 \end{smallmatrix}\right)\right)$ of $\mathscr{C}^{\textsf{pre-tr}}$ with $\mathrm{f} \in \mathscr{C}(\mathrm{X},\mathrm{Y})^0$.

Передтриангульовані dg-категорії

The category $\mathscr{C}$ is embedded into $\mathscr{C}^{\mathsf{pre\text{-}tr}}$ as a full differential graded subcategory via $\mathrm{X} \mapsto (\mathrm{X}[0], 0)$ and we identify $\mathscr{C}$ with its image.

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Definition

We say that a dg-category $\mathscr{C}$ is pretriangulated if every object X of $\mathscr{C}^{\textsf{pre-tr}}$ is isomorphic in $\mathrm{H}^0\mathscr{C}^{\textsf{pre-tr}}$ to some object Y of $\mathrm{H}^0\mathscr{C}$.

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The differential graded category $\mathscr{C}$ is pretriangulated if for every $X \in \mathsf{Ob}\,\mathscr{C}$, $k \in \mathbb{Z}$ the object $X[k]$ of $H^0(\mathscr{C}^{\mathsf{pre\text{-}tr}})$ is isomorphic to an object of $H^0(\mathscr{C})$ and for every closed morphism f in $\mathscr{C}$ of degree 0 the object $\mathsf{Cone}(f) \in \mathsf{Ob}\,\mathscr{C}^{\mathsf{pre\text{-}tr}}$ is isomorphic in $H^0(\mathscr{C}^{\mathsf{pre\text{-}tr}})$ to an object of $H^0(\mathscr{C})$. The first condition implies that $\mathscr{C}$ is closed under shifts.

Claim

Let $\mathscr{C}$ be a dg-category. Then the dg-category $\mathscr{C}^{\mathsf{pre\text{-}tr}}$ is pretriangulated. The dg-functors $u_{\mathsf{pre\text{-}tr}}, u_{\mathsf{pre\text{-}tr}}^{\mathsf{pre\text{-}tr}} : \mathscr{C}^{\mathsf{pre\text{-}tr}} \to \mathscr{C}^{\mathsf{pre\text{-}tr}\,\mathsf{pre\text{-}tr}}$ and $m_{\mathsf{pre\text{-}tr}} : \mathscr{C}^{\mathsf{pre\text{-}tr}\,\mathsf{pre\text{-}tr}} \to \mathscr{C}^{\mathsf{pre\text{-}tr}}$ are equivalences, quasi-inverse to each other.

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$\mathsf{Tr}(\mathscr{C}) = \mathrm{H}^0(\mathscr{C}^{\mathsf{pre\text{-}tr}})$, $\mathscr{C}^{\mathsf{tr}} = \mathrm{H}^{\bullet}(\mathscr{C}^{\mathsf{pre\text{-}tr}})$, $\mathsf{Pre\text{-}Tr}(\mathscr{C}) = \mathscr{C}^{\mathsf{pre\text{-}tr}}$.

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Definition 2. Let $\mathscr{A}$ be a DG-category, $C = \{E_i^{[n_i]}, q_{ij}\}$ and $C' = \{E_i'^{[n_i']}, q'_{ij}\}$ two objects in $\text{Pre-Tr}(\mathscr{A})$, and $f = \{f_{ij}\colon E_i \to E'_j\}$ a twisted morphism from C to C'. By the *cone* of this morphism we mean the object $\operatorname{Cone} f = \{E''_i, q''_{ij}\}$ for which

$$E''_i = E_i \oplus E'_{i-1}, \qquad q''_{ij} = \begin{pmatrix} q_{ij} & f_{ij} \\ 0 & q'_{ij} \end{pmatrix}.$$

We have in $\text{Pre-Tr}(\mathscr{A})$ the natural closed morphisms

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ & {\scriptstyle +1}\nwarrow \quad \swarrow & \\ & \operatorname{Cone} f & \end{array}, \tag{1.1}$$

inducing also morphisms in $\text{Tr}(\mathscr{A})$.

By the *distinguished triangles* in $\text{Tr}(\mathscr{A})$ we shall mean the triangles isomorphic to those of the form (1.1).

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plexes. The category of 0th cohomology of the DG-category $DG\text{-Fun}^0(\mathscr{A}, C(\mathscr{A}b))$ we denote by $\mathrm{Hot}(\mathscr{A})$. If $\mathscr{A}$ has a single object and trivial DG-structure, i.e., if it amounts simply to a ring, then $\mathrm{Hot}(\mathscr{A})$ is the homotopy category of complexes of modules over that ring.

Let $t\colon \varphi \to \psi$ be a DG-transformation of contravariant DG-functors $\mathscr{A} \to C(\mathscr{A}b)$. We define a new DG-functor, $\mathrm{Cone}(t)\colon \mathscr{A} \to C(\mathscr{A}b)$, assigning to an object $E \in \mathrm{Ob}\mathscr{A}$ the complex $\mathrm{Cone}\{t_E\colon \varphi(E) \to \psi(E)\}$.

We have in $DG\text{-Fun}^0(\mathscr{A}, C(\mathscr{A}b))$

$$\begin{array}{ccc} \varphi & \xrightarrow{t} & \psi \\ & {}_{+1}\nwarrow \quad \swarrow, & \\ & \mathrm{Cone}(t) & \end{array} \tag{1.2}$$

determining also a triangle in $\mathrm{Hot}(\mathscr{A})$.

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PROPOSITION 2. *The category* $\mathrm{Hot}(\mathscr{A})$ *with the above-described set of distinguished triangles and componentwise translation functor is a triangulated category.*

PROOF. The verification of Verdier's axioms TR1–TR4 for the case of the ordinary homotopy category of complexes, as carried out, e.g., in [2], applies without change to the present case, which is in essence the case of complexes with a system of operators (generating the DG-category).

Похідна категорія dg-модулів над dg-категорією

DEFINITION 3. Let $\mathscr{A}$ be a DG-category. We define an imbedding of DG-categories

$$\alpha\colon \text{Pre-Tr}(\mathscr{A}) \to DG\text{-Fun}^0(\mathscr{A}, C(\mathscr{A}b)).$$

The imbedding assigns to an object $K = \{E_i^{[n_i]}, q_{ij}\} \in \text{Ob}\,\text{Pre-Tr}(\mathscr{A})$ the following DG-functor $\alpha(K)\colon \mathscr{A} \to C(\mathscr{A}b)$. For each $E \in \text{Ob}\,\mathscr{A}$ the value $\alpha(K)(E)$ is the graded abelian group $\bigoplus \text{Hom}_{\mathscr{A}}(E, E_i)[n_i]$ provided with the differential $d+Q$, where $Q = \|q_{ij}\|$ and d is the differential in $\bigoplus \text{Hom}_{\mathscr{A}}(E, E_i)[n_i]$.

PROPOSITION 3. (a) *The functor* α *is an imbedding of* Pre-Tr($\mathscr{A}$) *into* DG-Fun$^0(\mathscr{A}, C(\mathscr{A}b))$ *as a full DG-subcategory, and it takes the cone of a closed morphism f in* Pre-Tr($\mathscr{A}$) *into the cone of the morphism $\alpha(f)$ in* DG-Fun$^0(\mathscr{A}, C(\mathscr{A}b))$.

(b) *The cohomology functor $H(\alpha)$ gives an imbedding of* Tr($\mathscr{A}$) *into* Hot($\mathscr{A}$) *as a full triangulated subcategory.*

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(b) *The cohomology functor* $H(\alpha)$ *gives an imbedding of* Tr$(\mathscr{A})$ *into* Hot$(\mathscr{A})$ *as a full triangulated subcategory.*

Proposition 1. *The category* Tr$(\mathscr{A})$ *with the above-described set of distinguished triangles is a triangulated category.*

$$\mathrm{D}(\mathscr{A}) = \mathsf{Hot}(\mathscr{A}) = \mathrm{DG\text{-}Fun}^{\circ}(\mathscr{A}, \mathrm{C}(\Bbbk\text{-}\mathsf{mod})).$$

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Proposition 1. *The category* Tr($\mathscr{A}$) *with the above-described set of distinguished triangles is a triangulated category.*

$$D(\mathscr{A}) = \text{Hot}(\mathscr{A}) = \text{DG-Fun}^\circ(\mathscr{A}, C(\Bbbk\text{-mod})).$$

We recall that a monad in a category $\mathscr{B}$ (see [3]) is a functor $C\colon \mathscr{B} \to \mathscr{B}$ together with natural transformations $\mu\colon C \circ C \to C$ and $\eta\colon \mathrm{id}_{\mathscr{B}} \to C$ such that for every $B \in \operatorname{Ob}\mathscr{B}$ the composite morphisms

$$C(B) \xrightarrow{\eta_{C(B)}} C(C(B)) \xrightarrow{\mu_B} C(B)$$

$$C(B) \xrightarrow{C(\eta_B)} C(C(B)) \xrightarrow{\mu_B} C(B) = \mathrm{id}$$

$$\begin{array}{ccc} C(C(C(B))) & \xrightarrow{\mu_{C(B)}} & C(C(B)) \\ {\scriptstyle C(\mu_B)}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ C(C(b)) & \xrightarrow{\mu_B} & C(B) \end{array}$$

commutes.

Let $\mathscr{B} = DG\text{-Cat}$ be the category of DG-categories, and $\mathscr{A} \in \operatorname{Ob}\mathscr{B}$ a given DG-category.

We construct a DG-functor

$$\mathrm{Tot}_{\mathscr{A}}\colon \text{Pre-Tr}(\text{Pre-Tr}(\mathscr{A})) \to \text{Pre-Tr}(\mathscr{A}).$$

Namely, an object in $\text{Pre-Tr}(\text{Pre-Tr}(\mathscr{A}))$ can be regarded as a set $C = \{(C_{ij})_{i,j\in\mathbf{Z}},\ q_{ij,kl}\colon C_{ij} \to C_{kl}\}$ with appropriate differential conditions on the $q_{ij,kl}$. Put

$$\mathrm{Tot}_{\mathscr{A}}(C) = \{(D_k)_{k\in\mathbf{Z}},\ r_{kl}\colon D_k \to D_l\},$$

where

$$D_k = \bigoplus_{i+j=k} C_{ij}, \qquad r_{kl} = \|q_{ij,mn}\|, \qquad i+j=k,\ m+n=l.$$

We shall call $\mathrm{Tot}_{\mathscr{A}}(C)$ the *convolution* of the twisted complex C over $\text{Pre-Tr}(\mathscr{A})$.

Clearly, the correspondence $\mathscr{A} \mapsto \mathrm{Tot}_{\mathscr{A}}$ extends to a natural transformation

$$\mathrm{Tot}\colon \text{Pre-Tr} \circ \text{Pre-Tr} \to \text{Pre-Tr}$$

on the category DG-Cat.

We denote by $\varepsilon_{\mathscr{A}}$ the natural imbedding of $\mathscr{A}$ into $\text{Pre-Tr}(\mathscr{A})$ as a full DG-subcategory: $\varepsilon_{\mathscr{A}}(E)$ is the set consisting of just E at the zeroth position. Thus, ε is a natural transformation $\mathrm{id} \to \text{Pre-Tr}$ on the category DG-Cat.

PROPOSITION 1. *The functor*

$$\text{Pre-Tr}\colon DG\text{-Cat} \to DG\text{-Cat}$$

and the natural transformations

$$\varepsilon\colon \mathrm{id} \to \text{Pre-Tr}, \qquad \mathrm{Tot}\colon \text{Pre-Tr}(\text{Pre-Tr}) \to \text{Pre-Tr}$$

define a monad over the category DG-Cat.

Напіввільна dg-категорія

Proposition 2. *The functor* $\mathrm{Tot}_{\mathscr{A}}$ *is an equivalence of DG-categories, and the cohomology functor*

$$H^0(\mathrm{Tot}_{\mathscr{A}})\colon \mathrm{Tr}(\text{Pre-Tr}(\mathscr{A})) \to \mathrm{Tr}(\mathscr{A})$$

is an equivalence of triangulated categories.

Proof. If C and C' are any two objects in $(\text{Pre-Tr})^2(\mathscr{A})$, then the complexes

$$\mathrm{Hom}_{(\text{Pre-Tr})^2(\mathscr{A})}(C, C') \quad \text{and} \quad \mathrm{Hom}_{\text{Pre-Tr}(\mathscr{A})}(\mathrm{Tot}\, C, \mathrm{Tot}\, C')$$

are the same. In other words, $\mathrm{Tot}_{\mathscr{A}}$ is an equivalence of DG-categories. Furthermore, by Proposition 1, it preserves convolutions of twisted complexes. Therefore $H^0(\mathrm{Tot}_{\mathscr{A}})$ is an equivalence of triangulated categories.

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Definition. Let $\mathcal{A}$ be a DG category $\mathcal{A}$ equipped with a DG functor $\mathcal{K} \to \mathcal{A}$. We say that $\mathcal{A}$ is *semi-free over* $\mathcal{K}$ if $\mathcal{A}$ can be represented as the union of an increasing sequence of DG subcategories $\mathcal{A}_i$, $i = 0, 1, \dots$, so that $\mathrm{Ob}\,\mathcal{A}_i = \mathrm{Ob}\,\mathcal{A}$, $\mathcal{K}$ maps isomorphically onto $\mathcal{A}_0$, and for every $i > 0$ $\mathcal{A}_i$ as a graded k-category over $\mathcal{A}_{i-1}$ (i.e., with forgotten differentials in the Hom complexes) is freely generated over $\mathcal{A}_{i-1}$ by a family of homogeneous morphisms f_α such that $\mathrm{d} f_\alpha \in \mathrm{Mor}\,\mathcal{A}_{i-1}$.

Definition. A DG category $\mathcal{A}$ is *semi-free* if it is semi-free over $\mathcal{A}_{\mathrm{discr}}$, where $\mathcal{A}_{\mathrm{discr}}$ is the DG category with $\mathrm{Ob}\,\mathcal{A}_{\mathrm{discr}} = \mathrm{Ob}\,\mathcal{A}$ such that the endomorphism DG algebra of each object of $\mathcal{A}_{\mathrm{discr}}$ equals k and $\mathrm{Hom}_{\mathcal{A}_{\mathrm{discr}}}(X, Y) = 0$ if X, Y are different objects of $\mathcal{A}_{\mathrm{discr}}$.

Квазіізоморфна напіввільна dg-категорія

B.5. Lemma. *For every DG category $\mathcal{A}$ there exists a semi-free DG category $\widetilde{\mathcal{A}}$ with* $\operatorname{Ob}\widetilde{\mathcal{A}} = \operatorname{Ob}\mathcal{A}$ *and a functor* $\Psi : \widetilde{\mathcal{A}} \to \mathcal{A}$ *such that* $\Psi(X) = X$ *for every* $X \in \operatorname{Ob}\widetilde{\mathcal{A}}$ *and* Ψ *induces a surjective quasi-isomorphism* $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(\Psi(X), \Psi(Y))$ *for every* $X, Y \in \widetilde{\mathcal{A}}$.

The proof is the same as for DG algebras [19, Sections 2, 4] and similar to that of Lemma B.3. $(\widetilde{\mathcal{A}}, \Psi)$ is constructed as the direct limit of $(\widetilde{\mathcal{A}}_i, \Psi_i)$ where $\operatorname{Ob}\widetilde{\mathcal{A}}_i = \operatorname{Ob}\mathcal{A}$, $\mathcal{A}_0 \hookrightarrow \mathcal{A}_1 \hookrightarrow \cdots$, $\Psi_i : \widetilde{\mathcal{A}}_i \to \mathcal{A}$, $\Psi_i|_{\widetilde{\mathcal{A}}_{i-1}} = \Psi_{i-1}$, and the following conditions are satisfied:

(i) $\mathcal{A}_0$ is the discrete k-category;

(ii) for every $i > 0$ $\mathcal{A}_i$ as a graded k-category is freely generated over $\mathcal{A}_{i-1}$ by a family of homogeneous morphisms f_α such that $\mathrm{d} f_\alpha \in \operatorname{Mor}\mathcal{A}_{i-1}$;

(iii) for every $i > 0$ and $X, Y \in \operatorname{Ob}\mathcal{A}$ the morphism $\operatorname{Hom}_{\mathcal{A}_i}(X, Y) \to \operatorname{Hom}_{\mathcal{A}}(\Psi(X), \Psi(Y))$ is surjective and induces a surjective map between the sets of the cocycles;

(iv) for every $i > 0$ and $X, Y \in \operatorname{Ob}\mathcal{A}$ every cocycle $f \in \operatorname{Hom}_{\mathcal{A}_i}(X, Y)$ whose image in $\operatorname{Hom}_{\mathcal{A}}(\Psi(X), \Psi(Y))$ is a coboundary becomes a coboundary in $\operatorname{Hom}_{\mathcal{A}_{i+1}}(X, Y)$.

Квазіізоморфна напіввільна dg-категорія

B.5. Lemma. *For every DG category $\mathcal{A}$ there exists a semi-free DG category $\widetilde{\mathcal{A}}$ with* $\operatorname{Ob}\widetilde{\mathcal{A}} = \operatorname{Ob}\mathcal{A}$ *and a functor* $\Psi : \widetilde{\mathcal{A}} \to \mathcal{A}$ *such that* $\Psi(X) = X$ *for every* $X \in \operatorname{Ob}\widetilde{\mathcal{A}}$ *and* Ψ *induces a surjective quasi-isomorphism* $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(\Psi(X), \Psi(Y))$ *for every* $X, Y \in \widetilde{\mathcal{A}}$.

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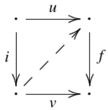
One constructs $(\widetilde{\mathcal{A}}_i, \Psi_i)$ by induction. Notice that (iii) holds for all i if it holds for $i = 1$, so after $(\widetilde{\mathcal{A}}_1, \Psi_1)$ is constructed one only has to kill cohomology classes by adding new morphisms.

Властивості підйому

Classically, **cofibrations** and **fibrations**, technical terms in the context of any model structure, refer to classes of continuous functions of topological spaces characterized by certain lifting properties. Our story begins by explaining the common features of any class of maps defined in this way. We will given an "algebraic" characterization of such classes in Chapter 12.

Let i and f be arrows in a fixed category $\mathcal{M}$. A **lifting problem** between i and f is simply a commutative square



A **lift** or **solution** is a dotted arrow, as indicated, making both triangles commute. If any lifting problem between i and f has a solution, we say that i has the **left lifting property** with respect to f and, equivalently, that f has the **right lifting property** with respect to i. We use the suggestive symbolic notation $i \boxslash f$ to encode these equivalent assertions.

Example 11.1.1. A map of sets has the right lifting property against the unique map $\emptyset \to *$ if and only if the map is an epimorphism. A map of sets has the right lifting property against the unique map $* \sqcup * \to *$ if and only if the map is a monomorphism.

Слабка система факторизації

Suppose $\mathcal{L}$ is a class of maps in $\mathcal{M}$. We write $\mathcal{L}^{\boxslash}$ for the class of arrows that have the right lifting property against each element of $\mathcal{L}$. Dually, we write ${}^{\boxslash}\mathcal{R}$ for the class of arrows that have the left lifting property against a given class $\mathcal{R}$.

EXAMPLE 11.1.3. Writing i_0 and p_0 for the obvious maps induced by the inclusion of the 0th endpoint of the standard unit interval I, the **Hurewicz fibrations** are defined to be $\{i_0 \colon A \to A \times I\}^{\boxslash}$ and the **Hurewicz cofibrations** are ${}^{\boxslash}\{p_0 \colon A^I \to A\}$, where the classes defining these lifting properties are indexed by all topological spaces A. Restricting to the subset of cylinder inclusions on disks, $\{i_0 \colon D^n \to D^n \times I\}^{\boxslash}$ is the class of **Serre fibrations**.

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In a model category, the lifting properties defining the cofibrations and fibrations are supplemented with a factorization axiom in the following manner:

Definition 11.2.1. A **weak factorization system** on a category is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

(factorization) every arrow can be factored as an arrow of $\mathcal{L}$ followed by an arrow of $\mathcal{R}$,

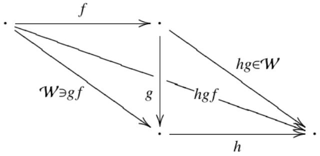
(lifting) $\mathcal{L} \boxslash \mathcal{R}$, and

(closure) furthermore, $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Модельні категорії

Definition 2.1.1. A **homotopical category** is a category $\mathcal{M}$ equipped with a wide[1] subcategory $\mathcal{W}$ such that for any composable triple of arrows

(2.1.2)



$$\Rightarrow \qquad f, g, h, hgf \in \mathcal{W}$$

if hg and gf are in $\mathcal{W}$ so are f, g, h, and hgf.

The arrows in $\mathcal{W}$ are called **weak equivalences**; the condition (2.1.2) is called the **2-of-6 property**.

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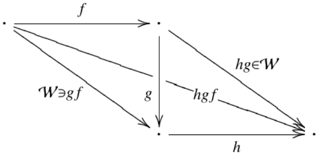
(2.1.2) 

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Quillen's **closed model categories** of [Qui67] are called **model categories** by the modern literature. Because a given category can admit multiple model category structures, we prefer to use the term **model structure** when referring to particular classes of maps that define a model category.

The following definition, perhaps first due to [JT07], a source of several useful facts about model categories, is equivalent to the usual one.

Definition 11.3.1. A **model structure** on a complete and cocomplete homotopical category $(\mathcal{M}, \mathcal{W})$ consists of two classes of morphisms $\mathcal{C}$ and $\mathcal{F}$ such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems.

The maps in $\mathcal{C}$ are called **cofibrations** and the maps in $\mathcal{F}$ are called **fibrations**. The maps in $\mathcal{C} \cap \mathcal{W}$ are called **trivial cofibrations** or **acyclic cofibrations** while the maps in $\mathcal{F} \cap \mathcal{W}$ are called **trivial fibrations** or **acyclic fibrations**. The model structure is said to be **cofibrantly generated** if both of its weak factorization systems are.

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We should at least state the upshot. A model structure on $(\mathcal{M}, \mathcal{W})$ in particular gives a notion of **fibrant** and **cofibrant** objects—more about which in just a moment. An object is fibrant just when the map to the terminal object is a fibration and cofibrant just when the map from the initial object is a cofibration. In a model category, it is an elementary exercise to show that

- every object is weakly equivalent to one that is both fibrant and cofibrant.

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Example 11.3.7. The category $\mathbf{Ch}_\bullet(R)$ of unbounded chain complexes of modules over a ring R has a model structure, due in this context to Mark Hovey [**Hov99**, §2.3], whose weak equivalences are quasi-isomorphisms and whose trivial fibrations and fibrations are defined by the lifting properties $\{S^{n-1} \to D^n \mid n \in \mathbb{Z}\}^{\boxslash}$ and $\{0 \to D^n \mid n \in \mathbb{Z}\}^{\boxslash}$. Here S^n is the chain complex with R in degree n and zeros elsewhere, and D^n has R in degrees n, $n-1$ with an identity differential.

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PROPOSITION 2.3.4. *A map $p\colon X \to Y$ in $\mathrm{Ch}(R)$ is a fibration if and only if $p_n\colon X_n \to Y_n$ is surjective for all n.*

PROOF. A diagram of the form

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ D^n & \longrightarrow & Y \end{array}$$

is equivalent to an element y in Y_n. A lift in this diagram is equivalent to an element x in X_n such that $px = y$. The lemma follows immediately. $\square$

Кофібрації для комплексів

Lemma 2.3.6. *Suppose R is a ring. If A is a cofibrant chain complex, then A_n is a projective R-module for all n. As a partial converse, any bounded below complex of projective R-modules is cofibrant.*

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Proposition 2.3.9. *Suppose R is a ring. Then a map $i\colon A \to B$ in $\mathrm{Ch}(R)$ is a cofibration if and only if i is a dimensionwise split injection with cofibrant cokernel.*

Припустимо, що $\alpha : \mathrm{M} \to \mathrm{N} \in \mathrm{dg}$. Позначимо через $\mathsf{Cone}\,\alpha = (\mathrm{M}[1] \oplus \mathrm{N}, \mathrm{d}_{\mathsf{Cone}}) \in \mathsf{Ob}\,\mathrm{dg}$ градуйований $\Bbbk$-модуль з диференціалом

$$\mathrm{d}_{\mathsf{Cone}} = \begin{pmatrix} \mathrm{d}_{\mathrm{M}}[1] & \sigma^{-1}\alpha \\ 0 & \mathrm{d}_{\mathrm{N}} \end{pmatrix} = \begin{pmatrix} -\sigma^{-1}\mathrm{d}_{\mathrm{M}}\sigma & \sigma^{-1}\alpha \\ 0 & \mathrm{d}_{\mathrm{N}} \end{pmatrix}.$$

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Наступний результат узагальнює теорему Хініча.

Theorem

Припустимо, що S - це множина, категорія $\mathscr{C}$ є повною і коповною і $\mathrm{F} : \mathrm{dg}^{\mathrm{S}} \rightleftarrows \mathscr{C} : \mathrm{U}$ є спряженням. Припустимо, що U зберігає фільтруючі кограниці. Для будь-якого $\mathrm{x} \in \mathrm{S}$ розглянемо об'єкт $\mathbb{K}_{\mathrm{x}}$ з dg^{S}, $\mathbb{K}_{\mathrm{x}}(\mathrm{x}) = \mathsf{Cone}(\mathsf{id}_{\Bbbk})$, $\mathbb{K}_{\mathrm{x}}(\mathrm{y}) = 0$ для $\mathrm{y} \neq \mathrm{x}$. Припустимо, що ланцюгове відображення $\mathrm{U}(\mathsf{in}_2) : \mathrm{UA} \to \mathrm{U}(\mathrm{F}(\mathbb{K}_{\mathrm{x}}[\mathrm{p}]) \sqcup \mathrm{A})$ - квазіізоморфізм для всіх об'єктів A з $\mathscr{C}$ і всіх $\mathrm{x} \in \mathrm{S}$, $\mathrm{p} \in \mathbb{Z}$. Оснастимо $\mathscr{C}$ класами слабких еквівалентів (відповідно фібрацій), що складаються з морфізмів f з $\mathscr{C}$ таких що Uf - квазіізоморфізм (відповідно епіморфізм). Тоді категорія $\mathscr{C}$ - модельна категорія.

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