3. dg-категорії.
Навколо похідних категорій

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$\mathscr{V}$-категорії

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Definition 3.3.1. A $\mathcal{V}$-**category** $\underline{\mathcal{D}}$ consists of

- a collection of objects $x, y, z \in \underline{\mathcal{D}}$
- for each pair $x, y \in \underline{\mathcal{D}}$, a **hom-object** $\underline{\mathcal{D}}(x, y) \in \mathcal{V}$
- for each $x \in \underline{\mathcal{D}}$, a morphism $\mathrm{id}_x\colon * \to \underline{\mathcal{D}}(x, x)$ in $\mathcal{V}$
- for each triple $x, y, z \in \underline{\mathcal{D}}$, a morphism $\circ\colon \underline{\mathcal{D}}(y, z) \times \underline{\mathcal{D}}(x, y) \to \underline{\mathcal{D}}(x, z)$ in $\mathcal{V}$

such that the following diagrams commute for all $x, y, z, w \in \underline{\mathcal{D}}$:

$$\begin{array}{ccc}
\underline{\mathcal{D}}(z,w) \times \underline{\mathcal{D}}(y,z) \times \underline{\mathcal{D}}(x,y) & \xrightarrow{1\times\circ} & \underline{\mathcal{D}}(z,w) \times \underline{\mathcal{D}}(x,z) \\
{\scriptstyle \circ\times 1}\downarrow & & \downarrow{\scriptstyle \circ} \\
\underline{\mathcal{D}}(y,w) \times \underline{\mathcal{D}}(x,y) & \xrightarrow{\circ} & \underline{\mathcal{D}}(x,w)
\end{array}$$

$$\begin{array}{ccc}
\underline{\mathcal{D}}(x,y) \times * & \xrightarrow{1\times \mathrm{id}_x} & \underline{\mathcal{D}}(x,y) \times \underline{\mathcal{D}}(x,x) \\
& {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \circ} \\
& & \underline{\mathcal{D}}(x,y)
\end{array}
\qquad
\begin{array}{ccc}
\underline{\mathcal{D}}(y,y) \times \underline{\mathcal{D}}(x,y) & \xleftarrow{\mathrm{id}_y\times 1} & * \times \underline{\mathcal{D}}(x,y) \\
{\scriptstyle \circ}\downarrow & \swarrow{\scriptstyle \cong} & \\
\underline{\mathcal{D}}(x,y) & &
\end{array}$$

Definition 1.3.1.1. Let k be a commutative ring. A *differential graded category* $\mathcal{C}$ over k consists of the following data:

- A collection $\{X, Y, \ldots\}$, whose elements are called the *objects* of $\mathcal{C}$.
- For every pair of objects X and Y, a chain complex of k-modules

$$\cdots \to \mathrm{Map}_{\mathcal{C}}(X,Y)_1 \to \mathrm{Map}_{\mathcal{C}}(X,Y)_0 \to \mathrm{Map}_{\mathcal{C}}(X,Y)_{-1} \to \cdots,$$

  which we will denote by $\mathrm{Map}_{\mathcal{C}}(X,Y)_*$.
- For every triple of objects X, Y, and Z, a composition map

$$\mathrm{Map}_{\mathcal{C}}(Y,Z)_* \otimes_k \mathrm{Map}_{\mathcal{C}}(X,Y)_* \to \mathrm{Map}_{\mathcal{C}}(X,Z)_*,$$

  which we can identify with a collection of k-bilinear maps

$$\circ : \mathrm{Map}_{\mathcal{C}}(Y,Z)_p \times \mathrm{Map}_{\mathcal{C}}(X,Y)_q \to \mathrm{Map}_{\mathcal{C}}(X,Z)_{p+q}$$

  satisfying the Leibniz rule $d(g \circ f) = dg \circ f + (-1)^p g \circ df$.
- For each object $X \in \mathcal{C}$, an *identity morphism* $\mathrm{id}_X \in \mathrm{Map}_{\mathcal{C}}(X,X)_0$ such that

$$g \circ \mathrm{id}_X = g \qquad \mathrm{id}_X \circ f = f$$

  for all $f \in \mathrm{Map}_{\mathcal{C}}(Y,X)_p$, $g \in \mathrm{Map}_{\mathcal{C}}(X,Y)_q$.

The composition law is required to be associative in the following sense: for every triple $f \in \mathrm{Map}_{\mathcal{C}}(W,X)_p$, $g \in \mathrm{Map}_{\mathcal{C}}(X,Y)_q$, and $h \in \mathrm{Map}_{\mathcal{C}}(Y,Z)_r$, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

In the special case where $k = \mathbf{Z}$ is the ring of integers, we will refer to a differential graded category over k simply as a *differential graded category*.

$\mathbf{dg}$-збагачення

Категорія $\mathbf{dg}$ – замкнена $\Rightarrow$ допускає збагачення в собі.
$\forall X, Y \in \mathbf{dg}$ $\underline{\mathbf{dg}}(X,Y) = \underline{\mathbf{dg}}^{I}(X,Y) \in \mathbf{dg}$;
З аксіоми для ev^{I} $\Rightarrow$ існує множення $\circ \in \mathbf{dg}$, асоціативне і унітальне.

dg-категорії

A DG category is an additive category $\mathcal{A}$ in which the sets $\mathrm{Hom}(A, B)$, $A, B \in \mathrm{Ob}\,\mathcal{A}$, are provided with a structure of a $\mathbb{Z}$-graded k-module and a differential $d : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B)$ of degree 1, so that for every $A, B, C \in \mathcal{A}$, the composition $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ comes from a morphism of complexes $\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$. Also there is a closed degree-zero morphism $1_A \in \mathrm{Hom}(A, A)$, which behaves as the identity under composition of morphisms.

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Using the supercommutativity isomorphism $S \otimes T \simeq T \otimes S$ in the category of DG k-modules, one defines for every DG category $\mathcal{A}$ the opposite DG category $\mathcal{A}^0$ with $\mathrm{Ob}\,\mathcal{A}^0 = \mathrm{Ob}\,\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}^0}(A, B) = \mathrm{Hom}_{\mathcal{A}}(B, A)$. We denote by $\mathcal{A}^{gr}$ the *graded* category which is obtained from $\mathcal{A}$ by forgetting the differentials on Hom's.

The tensor product of DG categories $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:

(i) $\mathrm{Ob}(\mathcal{A} \otimes \mathcal{B}) := \mathrm{Ob}\,\mathcal{A} \times \mathrm{Ob}\,\mathcal{B}$; for $A \in \mathrm{Ob}\,\mathcal{A}$ and $B \in \mathrm{Ob}\,\mathcal{B}$, the corresponding object is denoted by $A \otimes B$;

(ii) $\mathrm{Hom}(A \otimes B, A' \otimes B') := \mathrm{Hom}(A, A') \otimes \mathrm{Hom}(B, B')$ and the composition map is defined by $(f_1 \otimes g_1)(f_2 \otimes g_2) := (-1)^{\deg(g_1)\deg(f_2)} f_1 f_2 \otimes g_1 g_2$.

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Note that the DG categories $\mathcal{A} \otimes \mathcal{B}$ and $\mathcal{B} \otimes \mathcal{A}$ are canonically isomorphic. In the above notation, the isomorphism functor ϕ is

$$\phi(A \otimes B) = (B \otimes A), \qquad \phi(f \otimes g) = (-1)^{\deg(f)\deg(g)}(g \otimes f). \tag{4.1}$$

$\mathcal{V}$-функтори

Definition 3.5.1. A $\mathcal{V}$-**functor** $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ between $\mathcal{V}$-categories is given by an object map $\underline{\mathcal{C}} \ni x \mapsto Fx \in \underline{\mathcal{D}}$ together with morphisms

$$\underline{\mathcal{C}}(x,y) \xrightarrow{F_{x,y}} \underline{\mathcal{D}}(Fx,Fy)$$

in $\mathcal{V}$ for each $x,y \in \underline{\mathcal{C}}$ such that the following diagrams commute for all $x,y,z \in \underline{\mathcal{C}}$:

$$\begin{CD}
\underline{\mathcal{C}}(y,z) \times \underline{\mathcal{C}}(x,y) @>{\circ}>> \underline{\mathcal{C}}(x,z) \\
@V{F_{y,z}\times F_{x,y}}VV @VV{F_{x,z}}V \\
\underline{\mathcal{D}}(Fy,Fz) \times \underline{\mathcal{D}}(Fx,Fy) @>{\circ}>> \underline{\mathcal{D}}(Fx,Fz)
\end{CD}
\qquad
\begin{array}{ccc}
* & \xrightarrow{\mathrm{id}_x} & \underline{\mathcal{C}}(x,x) \\
& {\scriptstyle \mathrm{id}_{Fx}} \searrow & \downarrow {\scriptstyle F_{x,x}} \\
& & \underline{\mathcal{D}}(Fx,Fx)
\end{array}$$

dg-еквівалентності

Given a DG category $\mathcal{A}$, one defines the graded category $\mathrm{Ho}^{\cdot}(\mathcal{A})$ with $\mathrm{Ob}\,\mathrm{Ho}^{\cdot}(\mathcal{A}) = \mathrm{Ob}\,\mathcal{A}$ by replacing each Hom complex by the direct sum of its cohomology groups. We call $\mathrm{Ho}^{\cdot}(\mathcal{A})$ the *graded homotopy category* of $\mathcal{A}$. Restricting ourselves to the 0th cohomology of the Hom complexes, we get the *homotopy category* $\mathrm{Ho}(\mathcal{A})$.

Two objects $A, B \in \mathrm{Ob}\,\mathcal{A}$ are called DG *isomorphic* (or, simply, isomorphic) if there exists an invertible degree-zero morphism $f \in \mathrm{Hom}(A, B)$. We say that A, B are *homotopy-equivalent* if they are isomorphic in $\mathrm{Ho}(\mathcal{A})$.

A DG functor between DG categories $F : \mathcal{A} \to \mathcal{B}$ is said to be a *quasi-equivalence* if $\mathrm{Ho}^{\cdot}(F) : \mathrm{Ho}^{\cdot}(\mathcal{A}) \to \mathrm{Ho}^{\cdot}(\mathcal{B})$ is full and faithful and $\mathrm{Ho}(F)$ is essentially surjective. We say that F is a DG *equivalence* if it is full and faithful and every object of $\mathcal{B}$ is DG isomorphic to an object of $F(\mathcal{A})$. Certainly, a DG equivalence is a quasi-equivalence. DG categories $\mathcal{C}$ and $\mathcal{D}$ are called *quasi-equivalent* if there exist DG categories $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and a chain of quasi-equivalences

$$\mathcal{C} \longleftarrow \mathcal{A}_1 \longrightarrow \cdots \longleftarrow \mathcal{A}_n \longrightarrow \mathcal{D}. \tag{4.2}$$

Еквівалентності

Definition

A DG functor between DG categories $F : \mathscr{A} \to \mathscr{B}$ – еквівалентність = A_∞-еквівалентність, якщо

- $\forall X, Y \in \mathsf{Ob}\,\mathscr{A}$ ланцюгове відображення $F_{X,Y} : \mathscr{A}(X,Y) \to \mathscr{B}(XF, YF)$ гомотопічно оборотне, тобто, $[F_{X,Y}] \in H^0\underline{\mathbf{dg}}(\mathscr{A}(X,Y), \mathscr{B}(XF, YF))$ – ізоморфізм,
- $\mathrm{Ho}(F)$ is essentially surjective on objects, тобто, $\forall U \in \mathsf{Ob}\,\mathscr{B}$ $\exists V \in \mathsf{Ob}\,\mathscr{A}$ $\exists$ ізоморфізм $r \in H^0\mathscr{B}(U, VF)$.

Енд

The integral, called an end, is the limit of a particular diagram constructed from a functor that is both covariant and contravariant in $\mathscr{C}$. Given $H : \mathscr{C}^{\mathsf{op}} \times \mathscr{C} \to \mathscr{E}$, the end $\int_{\mathscr{C}} H$ is an object of $\mathscr{E}$ equipped with arrows $\int_{\mathscr{C}} H \to H(c,c)$ for each $c \in \mathscr{C}$ that are collectively universal with the property that the diagram

$$\begin{array}{ccc} \displaystyle\int_{\mathscr{C}} H & \longrightarrow & H(c',c') \\ \downarrow & & \downarrow{\scriptstyle H(f,c')} \\ H(c,c) & \xrightarrow{H(c,f)} & H(c,c') \end{array}$$

commutes for each $f : c \to c'$ in $\mathscr{C}$.

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commutes for each $f : c \to c'$ in $\mathscr{C}$.
Equivalently, $\int_{\mathscr{C}} H$ is the equalizer of the diagram

$$\int_{\mathscr{C}} H \dashrightarrow \prod_{c \in \mathsf{Ob}\mathscr{C}} H(c,c) \rightrightarrows \prod_{f \in \mathsf{Mor}\mathscr{C}} H(domf, codf).$$

Theorem 1.4.1. *Given functors $F, G : \mathcal{C} \to \mathcal{D}$ between small categories we have the canonical isomorphism of sets*

$$\mathrm{Cat}(\mathcal{C}, \mathcal{D})(F, G) \cong \int_C \mathcal{D}(FC, GC). \tag{1.47}$$

Proof A wedge $\tau_C : Y \to \mathcal{D}(FC, GC)$ consists of a function $y \mapsto (\tau_{C,y} : FC \to GC \mid C \in \mathcal{C})$, which is natural in $C \in \mathcal{C}$ (this is simply a rephrasing of the wedge condition): the equation

$$G(f) \circ \tau_{C,y} = \tau_{C',y} \circ F(f) \tag{1.48}$$

valid for any $f : C \to C'$, means that for a fixed $y \in Y$ the arrows τ_C form the components of a natural transformation $F \Rightarrow G$; thus, there exists a unique way to close the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tau_C} & \mathcal{D}(FC, GC) \\ & \searrow^{h} & \uparrow \\ & & \mathrm{Cat}(\mathcal{C}, \mathcal{D})(F, G) \end{array} \tag{1.49}$$

with a function sending $y \mapsto \tau_{\_,y} \in \prod_{C \in \mathcal{C}} \mathcal{D}(FC, GC)$, and where $\mathrm{Cat}(\mathcal{C}, \mathcal{D})(F, G) \to \mathcal{D}(FC, GC)$ is the wedge sending a natural transformation to its c-component; the diagram commutes for a single $h : Y \to \mathrm{Cat}(\mathcal{C}, \mathcal{D})(F, G)$, and this is precisely the desired universal property for $\mathrm{Cat}(\mathcal{C}, \mathcal{D})(F, G)$ to be $\int_C \mathcal{D}(FC, GC)$. $\square$

Перетворення

Given DG categories $\mathcal{A}$ and $\mathcal{B}$, the collection of covariant DG functors $\mathcal{A} \to \mathcal{B}$ is itself the collection of objects of a DG category, which we denote by $\mathrm{Fun}_{\mathrm{DG}}(\mathcal{A}, \mathcal{B})$. Namely, let ϕ and ψ be two DG functors. Put $\mathrm{Hom}^k(\phi, \psi)$ equal to the set of natural transformations $t : \phi^{gr} \to \psi^{gr}[k]$ of graded functors from $\mathcal{A}^{gr}$ to $\mathcal{B}^{gr}$. This means that for any morphism $f \in \mathrm{Hom}^s_{\mathcal{A}}(A, B)$ one has

$$\psi(f) \cdot t(A) = (-1)^{ks} t(B) \cdot \phi(f). \tag{4.3}$$

On each $A \in \mathcal{A}$, the differential of the transformation t is equal to $(dt)(A)$ (one easily checks that this is well defined). Thus, the closed transformations of degree zero are the DG transformations of DG functors. A similar definition gives us the DG category consisting of the contravariant DG functors $\mathrm{Fun}_{\mathrm{DG}}(\mathcal{A}^0, \mathcal{B}) = \mathrm{Fun}_{\mathrm{DG}}(\mathcal{A}, \mathcal{B}^0)$ from $\mathcal{A}$ to $\mathcal{B}$.

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$$\begin{array}{ccc} \phi(A) & \xrightarrow{\phi(f)} & \phi(B) \\ {\scriptstyle t(A)}\downarrow & (-1)^{tf} & \downarrow{\scriptstyle t(B)} \\ \psi(A) & \xrightarrow{\psi(f)} & \psi(B) \end{array}$$

$$t = \{t(A) \mid A \in \mathrm{Ob}\,\mathscr{A}\}$$

$$t(A) \in \mathscr{B}(\phi(A), \psi(A))^k$$

$$\forall f \in \mathscr{A}(A, B)^s$$

Modulo size issues, the category $\mathcal{V}$-**Cat** is closed: given $\mathcal{V}$-categories $\underline{\mathcal{D}}$ and $\underline{\mathcal{M}}$ where $\underline{\mathcal{D}}$ is small, define a $\mathcal{V}$-category $\underline{\mathcal{M}}^{\underline{\mathcal{D}}}$ whose objects are $\mathcal{V}$-functors $F, G\colon \underline{\mathcal{D}} \rightrightarrows \underline{\mathcal{M}}$ and whose hom-objects, taking inspiration from 1.2.8, are defined by the formula

$$\underline{\mathcal{M}}^{\underline{\mathcal{D}}}(F,G) = \int_{d\in\underline{\mathcal{D}}} \underline{\mathcal{M}}(Fd, Gd). \tag{7.3.2}$$

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the formula for the enriched end defining the hom-object of $\mathcal{V}$-natural transformations is the equalizer

$$\int_{\underline{\mathcal{D}}} \underline{\mathcal{M}}(Fd, Gd) := \mathrm{eq}\left(\prod_{d} \underline{\mathcal{M}}(Fd, Gd) \rightrightarrows \prod_{d,d'} \underline{\mathcal{V}}(\underline{\mathcal{D}}(d,d'), \underline{\mathcal{M}}(Fd, Gd')) \right). \tag{7.3.3}$$

The component of the top arrow indexed by the ordered pair d, d' projects to $\underline{\mathcal{M}}(Fd, Gd)$ and composes in the second coordinate with $\underline{\mathcal{D}}(d, d')$. The analogous component of the bottom arrow projects to $\underline{\mathcal{M}}(Fd', Gd')$ and precomposes in the first coordinate with $\underline{\mathcal{D}}(d, d')$.

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Верхня стрілка в (7.3.3) – це

$$\prod_d \underline{\mathscr{M}}(Fd, Gd) \xrightarrow{\mathrm{pr}_d} \underline{\mathscr{M}}(Fd, Gd) \to \underline{\mathscr{V}}^l(\underline{\mathscr{D}}(d,d'), \underline{\mathscr{M}}(Fd, Gd')),$$

де останній морфізм приходить з парування

$$\underline{\mathscr{M}}(Fd, Gd) \otimes \underline{\mathscr{D}}(d,d') \xrightarrow{1\otimes G} \underline{\mathscr{M}}(Fd, Gd) \otimes \underline{\mathscr{M}}(Gd, Gd') \dot{\to} \underline{\mathscr{M}}(Fd, Gd').$$

Нижня стрілка в (7.3.3) – це

$$\prod_d \underline{\mathscr{M}}(Fd, Gd) \xrightarrow{\mathrm{pr}_{d'}} \underline{\mathscr{M}}(Fd', Gd') \to \underline{\mathscr{V}}^{l}(\underline{\mathscr{D}}(d, d'), \underline{\mathscr{M}}(Fd, Gd')),$$

де останній морфізм приходить з композиції

$$\begin{array}{ccc}
\underline{\mathscr{M}}(Fd', Gd') \otimes \underline{\mathscr{D}}(d, d') & \xrightarrow{\qquad\qquad} & \underline{\mathscr{M}}(Fd, Gd') \\
{\scriptstyle c}\big\downarrow & \nearrow & \big\uparrow{\scriptstyle \cdot} \\
\underline{\mathscr{D}}(d, d') \otimes \underline{\mathscr{M}}(Fd', Gd') & \xrightarrow{F \otimes 1} & \underline{\mathscr{M}}(Fd, Fd') \otimes \underline{\mathscr{M}}(Fd', Gd')
\end{array}$$

тоді як діагональній стрілці відповідало б

$$\underline{\mathscr{M}}(Fd', Gd') \to \underline{\mathscr{V}}^{r}(\underline{\mathscr{D}}(d, d'), \underline{\mathscr{M}}(Fd, Gd')).$$

dg-модулі над dg-категорією

We denote the DG category $\mathrm{Fun}_{DG}(\mathcal{B}, DG(k))$ by $\mathcal{B}$-mod and call it the category of DG $\mathcal{B}$-modules. There is a natural covariant DG functor $h : \mathcal{A} \to \mathcal{A}^0$-mod (the Yoneda embedding) defined by $h^A(B) := \mathrm{Hom}_{\mathcal{A}}(B, A)$. As in the "classical" case one verifies that the functor h is full and faithful, that is,

$$\mathrm{Hom}_{\mathcal{A}}(A, A') = \mathrm{Hom}_{\mathcal{A}^0\text{-mod}}(h^A, h^{A'}). \tag{4.4}$$

Moreover, for any $F \in \mathcal{A}^0$-mod, $A \in \mathcal{A}$,

$$\mathrm{Hom}_{\mathcal{A}^0\text{-mod}}(h^A, F) = F(A). \tag{4.5}$$

The $\mathcal{A}^0$-DG-modules h^A, $A \in \mathcal{A}$, are called *free*. An $\mathcal{A}^0$-DG-module F is called *semifree* if it has a filtration

$$0 = F_0 \subset F_1 \subset \cdots = F \tag{4.6}$$

such that F_{i+1}/F_i is isomorphic to a direct sum of shifted free $\mathcal{A}^0$-DG-modules $h^A[n]$, $n \in \mathbb{Z}$. The full subcategory of semifree $\mathcal{A}^0$-DG-modules is denoted by $SF(\mathcal{A})$.

An $\mathcal{A}^0$-DG-module F is called acyclic if the complex $F(A)$ is acyclic for all $A \in \mathcal{A}$. Let $D(\mathcal{A})$ denote the *derived category* of $\mathcal{A}^0$-DG-modules, that is, $D(\mathcal{A})$ is the Verdier quotient of the homotopy category $Ho(\mathcal{A}^0\text{-mod})$ by the subcategory of acyclic DG-modules.

dg-категорії з формальним зсувом

Given a DG category $\mathcal{A}$, one can associate to it a triangulated category $\mathcal{A}^{\mathrm{tr}}$ [5]. It is defined as the homotopy category of a certain DG category $\mathcal{A}^{\text{pre-tr}}$. The idea of the definition of $\mathcal{A}^{\text{pre-tr}}$ is to formally add cones of all morphisms, cones of morphisms between cones, and so forth.

First, we need to clarify the notion of a "formal shift" of an object.

Definition 4.5. Define the DG category $\bar{\mathcal{A}}$ as follows:

$$\operatorname{Ob}\bar{\mathcal{A}} = \{A[n] \mid A \in \operatorname{Ob}\mathcal{A},\ n \in \mathbb{Z}\}, \tag{4.12}$$

and define

$$\operatorname{Hom}_{\bar{\mathcal{A}}}\left(A[k], B[n]\right) = \operatorname{Hom}_{\mathcal{A}}(A, B)[n-k] \tag{4.13}$$

as *graded vector spaces*. If $f \in \operatorname{Hom}_{\mathcal{A}}(A, B)$ is considered as an element of $\operatorname{Hom}_{\bar{\mathcal{A}}}(A[k], B[n])$ under the above identification, then the differentials are related by the formula

$$d_{\bar{\mathcal{A}}}(f) = (-1)^n d_{\mathcal{A}}(f). \tag{4.14}$$

Notice, for example, that the differential in $\operatorname{Hom}_{\bar{\mathcal{A}}}(A[1], B[1])$ is equal to *minus* the differential in $\operatorname{Hom}_{\mathcal{A}}(A, B)$.

Алгебра в категорії **dg**-категорій

The category of differential graded categories $\mathbf{dg}\text{-}\mathscr{C}\mathit{at}$ equipped with the tensor product $\boxtimes$ is a symmetric monoidal category. Let us study an algebra in this category, which will be used to define the functor of shifts. Let $\mathscr{Z}$ be a differential graded quiver with $\mathsf{Ob}\,\mathscr{Z} = \mathbb{Z}$, $\mathscr{Z}(m,n) = \Bbbk[n-m]$ and zero differential.

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$$\begin{aligned} \mu' = \phi'(l,m,n) : \mathscr{Z}(l,m) \otimes_{\Bbbk} \mathscr{Z}(m,n) \\ = \Bbbk[m-l] \otimes \Bbbk[n-m] \to \Bbbk[n-l] = \mathscr{Z}(l,n), \\ 1s^{m-l} \otimes 1s^{n-m} \mapsto \phi'(l,m,n)s^{n-l}. \end{aligned}$$

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It is specified by a function $\phi' : \mathbb{Z}^3 \to \Bbbk^\times$ with values in the multiplicative group $\Bbbk^\times$. The associativity of composition

$$(1 \otimes \mu')\mu' = (\mu' \otimes 1)\mu' : \mathscr{Z}(k,l) \otimes \mathscr{Z}(l,m) \otimes \mathscr{Z}(m,n) \to \mathscr{Z}(k,n)$$

implies that ϕ' is a cycle:

$$\phi'(l,m,n) \cdot \phi'(k,l,n) = \phi'(k,l,m) \cdot \phi'(k,m,n).$$

An arbitrary such cycle ϕ' is a boundary

$$\phi'(l,m,n) = \xi(l,m)\cdot\xi(m,n)\cdot\xi(l,n)^{-1} \tag{1}$$

of a function $\xi : \mathbb{Z}^2 \to \Bbbk^\times$, namely, $\xi(l,m) = \phi'(0,l,m)$.

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The graded quiver automorphism $\xi : \mathscr{Z} \to \mathscr{Z}$, $n \mapsto n$, $\xi(m,n) : \mathscr{Z}(m,n) \to \mathscr{Z}(m,n)$ equips $\mathscr{Z}$ with another (isomorphic) category structure $(\mathscr{Z},\mu)$ with $\phi(l,m,n) = 1$, so that $\xi : (\mathscr{Z},\mu') \to (\mathscr{Z},\mu)$ is a category isomorphism.

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In the following we shall consider only composition μ in $\mathscr{Z}$ specified by the function $\phi(l,m,n) = 1$. The elements $1 \in \Bbbk = \mathscr{Z}(n,n)$ are identity morphisms of $\mathscr{Z}$.

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We equip the object $\mathscr{Z}$ of $(\mathbf{dg}\text{-}\mathscr{C}at,\boxtimes)$ with an algebra structure, given by multiplication – differential graded functor

$$\otimes_\psi : \mathscr{Z}\boxtimes\mathscr{Z} \to \mathscr{Z}, \qquad m\times n \mapsto m+n,$$

$$\begin{aligned}\otimes_\psi = \psi(n,m,k,l) : (\mathscr{Z}\boxtimes\mathscr{Z})(n\times m, k\times l)&\\ = \mathscr{Z}(n,k)\otimes\mathscr{Z}(m,l) &\to \mathscr{Z}(n+m,k+l),\\ 1s^{k-n}\otimes 1s^{l-m} &\mapsto \psi(n,m,k,l)s^{k+l-n-m}.\end{aligned}$$

We assume that the function $\psi : \mathbb{Z}^4 \to \Bbbk$ takes values in $\Bbbk^\times$.

Being a functor, $\otimes_\psi$ has to satisfy the equation:

$$\psi(a,b,c,d)\cdot\psi(c,d,e,f) = (-1)^{(d-b)(e-c)}\psi(a,b,e,f).$$

It specifies the boundary of the 2-cochain $\psi : \mathbb{Z}^4 \to \mathbb{k}^\times$. Generic solution to this equation is

$$\psi(a,b,c,d) = (-1)^{c(b-d)}\chi(a,b)\cdot\chi(c,d)^{-1} \tag{2}$$

for some function $\chi : \mathbb{Z}^2 \to \mathbb{k}^\times$ (take $\chi(a,b) = \psi(a,b,0,0)$).

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for some function $\chi:\mathbb{Z}^2\to\Bbbk^\times$ (take $\chi(a,b)=\psi(a,b,0,0)$). Associativity of the algebra $(\mathscr{Z},\otimes_\psi)$

$$\begin{array}{ccc}
(\mathscr{Z}\boxtimes\mathscr{Z}\boxtimes\mathscr{Z})(a\times b\times c, d\times e\times f) & \xrightarrow{1\boxtimes\otimes_\psi} & (\mathscr{Z}\boxtimes\mathscr{Z})(a\times(b+c), d\times(e+f)) \\
{\scriptstyle \otimes_\psi\boxtimes 1}\big\downarrow & = & \big\downarrow{\scriptstyle \otimes_\psi} \\
(\mathscr{Z}\boxtimes\mathscr{Z})((a+b)\times c, (d+e)\times f) & \xrightarrow{\quad\otimes_\psi\quad} & \mathscr{Z}(a+b+c, d+e+f)
\end{array}$$

is expressed by the equation

$$\psi(a,b,d,e)\cdot\psi(a+b,c,d+e,f) = \psi(b,c,e,f)\cdot\psi(a,b+c,d,e+f).$$

It means that the function $\overline{\psi} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \Bbbk^\times$, $\overline{\psi}(a,b,c,d) = \psi(a,c,b,d)$ is a 2-cocycle on the group $\mathbb{Z}^2$. For a graded category $\mathscr{Z}$ with multiplication (1) related to $(\mathscr{Z},\mu)$ via the category isomorphism $\xi : (\mathscr{Z},\mu') \to (\mathscr{Z},\mu)$, $n \mapsto n$, $\xi(m,n) : \mathscr{Z}(m,n) \to \mathscr{Z}(m,n)$, we would get the 2-cocycle

$$\overline{\psi}'(a,b,c,d) = \xi(a,b) \cdot \xi(c,d) \cdot \overline{\psi}(a,b,c,d) \cdot \xi(a+c,b+d)^{-1},$$

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Plugging (2) into equation for ψ, we reduce the latter to

$$\begin{aligned}\chi(a,b) \cdot \chi(a,b+c)^{-1} \cdot \chi(a+b,c) \cdot \chi(b,c)^{-1} \\ = \chi(d,e) \cdot \chi(d,e+f)^{-1} \cdot \chi(d+e,f) \cdot \chi(e,f)^{-1}.\end{aligned}$$

The common value of the left and right hand sides does not depend on the arguments $a,b,c,d,e,f \in \mathbb{Z}$. Setting $d = e = f = 0$ we find that this constant equals 1. Thus, $\chi : \mathbb{Z}^2 \to \Bbbk^\times$ is a 2-cocycle on the group $\mathbb{Z}$ with values in $\Bbbk^\times$. Since $\mathbb{Z}$ is a free group, its cohomology group $H^2(\mathbb{Z},\Bbbk^\times)$ vanishes.

Therefore, the cocycle χ has the form

$$\chi(a,b) = \lambda(a)\cdot\lambda(b)\cdot\lambda(a+b)^{-1}$$

for some function $\lambda : \mathbb{Z} \to \Bbbk^{\times}$. The corresponding function $\psi : \mathbb{Z}^4 \to \Bbbk^{\times}$ is given by the formula

$$\begin{aligned} &\psi_\lambda(a,b,c,d) \\ &= (-1)^{c(b-d)}\lambda(a)\cdot\lambda(b)\cdot\lambda(a+b)^{-1}\cdot\lambda(c)^{-1}\cdot\lambda(d)^{-1}\cdot\lambda(c+d). \end{aligned}$$

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Denote the multiplication functor $\otimes_{\psi_\lambda}$ also by $\otimes^\lambda$. Define an automorphism of the graded category $\mathscr{Z}$ as $\overline{\lambda} : \mathscr{Z} \to \mathscr{Z}$, $n \mapsto n$, $\overline{\lambda} = \lambda(m)^{-1} \cdot \lambda(n) : \mathscr{Z}(m,n) \to \mathscr{Z}(m,n)$. $\overline{\lambda} : (\mathscr{Z}, \otimes^1) \to (\mathscr{Z}, \otimes^\lambda)$ is an algebra isomorphism, where the first algebra uses

$$\lambda_1(a) = 1, \qquad \psi_1(a,b,c,d) = (-1)^{c(b-d)}.$$

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$$\lambda_1(a) = 1, \qquad \psi_1(a,b,c,d) = (-1)^{c(b-d)}.$$

The multiplicative cocycle $\overline{\psi}_1 \in Z^2(\mathbb{Z}^2,\Bbbk^{\times})$ comes via the homomorphism $\mathbb{Z}/2 \to \Bbbk^{\times}$, $a \mapsto (-1)^a$, from the additive cocycle $\mathbb{Z}^2\times\mathbb{Z}^2 \to \mathbb{Z}/2$, $(a,c,b,d) \mapsto c(b-d) \pmod 2$, which represents the only non-trivial element of the cohomology group $H^2(\mathbb{Z}^2,\mathbb{Z}/2) = \mathbb{Z}/2$ with values in the trivial $\mathbb{Z}^2$-module $\mathbb{Z}/2$.

In the following we shall consider only multiplication $\otimes_{\mathscr{Z}} = \otimes^1$ in $\mathscr{Z}$ specified by the function $\psi_1(a,b,c,d) = (-1)^{c(b-d)}$. Clearly, the algebra $(\mathscr{Z}, \otimes_{\mathscr{Z}})$ is unital with the unit $\eta_{\mathscr{Z}} : \mathbb{1} \to \mathscr{Z}$, $* \mapsto 0$, $\mathrm{id} : \mathbb{1}(*,*) = \Bbbk \to \mathscr{Z}(0,0)$.

Функтор зсувів

Given a **dg**-category $\mathscr{A}$, we produce another one $\mathscr{A}^{[]} = \mathscr{A} \boxtimes \mathscr{Z}$, obtained by adding formal shifts of objects. The set of objects is

$$\mathsf{Ob}\,\mathscr{A}^{[]} = \{X[n] = (X,n) \mid X \in \mathsf{Ob}\,\mathscr{A}, n \in \mathbb{Z}\} = \mathsf{Ob}\,\mathscr{A} \times \mathbb{Z}.$$

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The graded $\Bbbk$-module of morphisms is $\mathscr{A}^{[]}(X[n], Y[m]) = \mathscr{A}(X,Y) \otimes \Bbbk[m-n]$. We identify it with $\mathscr{A}(X,Y)[m-n]$. Given a morphism of **dg**-categories $f : \mathscr{A} \to \mathscr{B}$, we define another one $f^{[]} = f \boxtimes 1_{\mathscr{Z}} : \mathscr{A}^{[]} \to \mathscr{B}^{[]}$. On objects it acts as $\mathsf{Ob}\, f^{[]} : X[n] \mapsto (Xf)[n]$. Commutative diagram

$$\begin{array}{ccccc}
\mathscr{A}(X,Y)[m-n] & \xrightarrow{s^{n-m}} & \mathscr{A}(X,Y) \otimes \Bbbk & \xrightarrow{1 \otimes s^{m-n}} & \mathscr{A}(X,Y) \otimes \mathscr{Z}(n,m) \\
\downarrow{\scriptstyle f^{[m-n]}} & & \downarrow{\scriptstyle f \otimes 1} & & \downarrow{\scriptstyle f \otimes 1} \\
\mathscr{B}^{[]}(Xf,Yf)[m-n] & \xrightarrow{s^{n-m}} & \mathscr{B}(Xf,Yf) \otimes \Bbbk & \xrightarrow{1 \otimes s^{m-n}} & \mathscr{B}(Xf,Yf) \otimes \mathscr{Z}(n,m)
\end{array}$$

describes the action of $-^{[]}$ on morphisms in another presentation:

$$f^{[m-n]} = s^{n-m} f s^{m-n} : \mathscr{A}(X,Y)[m-n] \to \mathscr{B}(Xf,Yf)[m-n].$$

dg-категорії замкнені відносно зсувів

Диференціал на $\mathscr{A}^{[]}$ обчислюється як

$$d^{[]} = (-1)^{m-n} s^{n-m} d s^{m-n} : \mathscr{A}^{[]}(X[n], Y[m]) = \mathscr{A}(X, Y)[m-n] \\ \to \mathscr{A}(X, Y)[m-n] = \mathscr{A}^{[]}(X[n], Y[m]).$$

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Композиція для $\mathscr{A}^{[]}$ обчислюється як

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Definition

We say that a **dg**-category $\mathscr{C}$ is closed under shifts if every object $X[n]$ of $\mathscr{C}^{[]}$ is homotopy isomorphic in $\mathscr{C}^{[]}$ to some object $Y[0]$, $Y = [X, n] \in \mathrm{Ob}\,\mathscr{C}$.

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