

# 3. dg-категорії. Навколо похідних категорій

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## $\mathcal{V}$ -категории

enriched should be a **symmetric monoidal category**  $(\mathcal{V}, \times, *)$ . Here,  $\mathcal{V}$  is an ordinary category,  $- \times -: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$  is a bifunctor called the **monoidal product**, and  $* \in \mathcal{V}$  is called the **unit object**. We write “ $\times$ ” for the monoidal product because it will be the car-

$$\mathcal{V} = \mathbf{dg}, \times = \otimes = \otimes_{\mathbb{k}}, * = \mathbf{1}\mathbf{l} = \mathbb{k}.$$

DEFINITION 3.3.1. A  $\mathcal{V}$ -category  $\underline{\mathcal{D}}$  consists of

- a collection of objects  $x, y, z \in \underline{\mathcal{D}}$
- for each pair  $x, y \in \underline{\mathcal{D}}$ , a **hom-object**  $\underline{\mathcal{D}}(x, y) \in \mathcal{V}$
- for each  $x \in \underline{\mathcal{D}}$ , a morphism  $\text{id}_x: * \rightarrow \underline{\mathcal{D}}(x, x)$  in  $\mathcal{V}$
- for each triple  $x, y, z \in \underline{\mathcal{D}}$ , a morphism  $\circ: \underline{\mathcal{D}}(y, z) \times \underline{\mathcal{D}}(x, y) \rightarrow \underline{\mathcal{D}}(x, z)$  in  $\mathcal{V}$

such that the following diagrams commute for all  $x, y, z, w \in \underline{\mathcal{D}}$ :

$$\begin{array}{ccc} \underline{\mathcal{D}}(z, w) \times \underline{\mathcal{D}}(y, z) \times \underline{\mathcal{D}}(x, y) & \xrightarrow{1 \times \circ} & \underline{\mathcal{D}}(z, w) \times \underline{\mathcal{D}}(x, z) \\ \circ \times 1 \downarrow & & \downarrow \circ \\ \underline{\mathcal{D}}(y, w) \times \underline{\mathcal{D}}(x, y) & \xrightarrow{\circ} & \underline{\mathcal{D}}(x, w) \end{array}$$

$$\begin{array}{ccc} \underline{\mathcal{D}}(x, y) \times * & \xrightarrow{1 \times \text{id}_x} & \underline{\mathcal{D}}(x, y) \times \underline{\mathcal{D}}(x, x) & \quad \quad \quad \underline{\mathcal{D}}(y, y) \times \underline{\mathcal{D}}(x, y) & \xleftarrow{\text{id}_y \times 1} & * \times \underline{\mathcal{D}}(x, y) \\ & \searrow \cong & \downarrow \circ & & \downarrow \circ & \swarrow \cong \\ & & \underline{\mathcal{D}}(x, y) & & \underline{\mathcal{D}}(x, y) & \end{array}$$

**Definition 1.3.1.1.** Let  $k$  be a commutative ring. A *differential graded category*  $\mathcal{C}$  over  $k$  consists of the following data:

- A collection  $\{X, Y, \dots\}$ , whose elements are called the *objects* of  $\mathcal{C}$ .
- For every pair of objects  $X$  and  $Y$ , a chain complex of  $k$ -modules

$$\cdots \rightarrow \text{Map}_{\mathcal{C}}(X, Y)_1 \rightarrow \text{Map}_{\mathcal{C}}(X, Y)_0 \rightarrow \text{Map}_{\mathcal{C}}(X, Y)_{-1} \rightarrow \cdots,$$

which we will denote by  $\text{Map}_{\mathcal{C}}(X, Y)_*$ .

- For every triple of objects  $X$ ,  $Y$ , and  $Z$ , a composition map

$$\text{Map}_{\mathcal{C}}(Y, Z)_* \otimes_k \text{Map}_{\mathcal{C}}(X, Y)_* \rightarrow \text{Map}_{\mathcal{C}}(X, Z)_*,$$

which we can identify with a collection of  $k$ -bilinear maps

$$\circ : \text{Map}_{\mathcal{C}}(Y, Z)_p \times \text{Map}_{\mathcal{C}}(X, Y)_q \rightarrow \text{Map}_{\mathcal{C}}(X, Z)_{p+q}$$

satisfying the Leibniz rule  $d(g \circ f) = dg \circ f + (-1)^p g \circ df$ .

- For each object  $X \in \mathcal{C}$ , an *identity morphism*  $\text{id}_X \in \text{Map}_{\mathcal{C}}(X, X)_0$  such that

$$g \circ \text{id}_X = g \quad \text{id}_X \circ f = f$$

for all  $f \in \text{Map}_{\mathcal{C}}(Y, X)_p$ ,  $g \in \text{Map}_{\mathcal{C}}(X, Y)_q$ .

The composition law is required to be associative in the following sense: for every triple  $f \in \text{Map}_{\mathcal{C}}(W, X)_p$ ,  $g \in \text{Map}_{\mathcal{C}}(X, Y)_q$ , and  $h \in \text{Map}_{\mathcal{C}}(Y, Z)_r$ , we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

In the special case where  $k = \mathbf{Z}$  is the ring of integers, we will refer to a differential graded category over  $k$  simply as a *differential graded category*.

## **dg**-збагачення

Категорія **dg** – замкнена  $\Rightarrow$  допускає збагачення в собі.

$$\forall X, Y \in \mathbf{dg} \quad \underline{\mathbf{dg}}(X, Y) = \underline{\mathbf{dg}}^l(X, Y) \in \mathbf{dg};$$

З аксіоми для  $\text{ev}^l \Rightarrow$  існує множення  $\circ \in \mathbf{dg}$ , асоціативне і унітальне.

## **dg**-категории

A DG category is an additive category  $\mathcal{A}$  in which the sets  $\text{Hom}(A, B)$ ,  $A, B \in \text{Ob } \mathcal{A}$ , are provided with a structure of a  $\mathbb{Z}$ -graded  $k$ -module and a differential  $d : \text{Hom}(A, B) \rightarrow \text{Hom}(A, B)$  of degree 1, so that for every  $A, B, C \in \mathcal{A}$ , the composition  $\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$  comes from a morphism of complexes  $\text{Hom}(A, B) \otimes \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$ . Also there is a closed degree-zero morphism  $1_A \in \text{Hom}(A, A)$ , which behaves as the identity under composition of morphisms.

Using the supercommutativity isomorphism  $S \otimes T \simeq T \otimes S$  in the category of DG  $k$ -modules, one defines for every DG category  $\mathcal{A}$  the opposite DG category  $\mathcal{A}^0$  with  $\text{Ob } \mathcal{A}^0 = \text{Ob } \mathcal{A}$ ,  $\text{Hom}_{\mathcal{A}^0}(A, B) = \text{Hom}_{\mathcal{A}}(B, A)$ . We denote by  $\mathcal{A}^{\text{gr}}$  the *graded* category which is obtained from  $\mathcal{A}$  by forgetting the differentials on Hom's.

The tensor product of DG categories  $\mathcal{A}$  and  $\mathcal{B}$  is defined as follows:

- (i)  $\text{Ob}(\mathcal{A} \otimes \mathcal{B}) := \text{Ob } \mathcal{A} \times \text{Ob } \mathcal{B}$ ; for  $A \in \text{Ob } \mathcal{A}$  and  $B \in \text{Ob } \mathcal{B}$ , the corresponding object is denoted by  $A \otimes B$ ;
- (ii)  $\text{Hom}(A \otimes B, A' \otimes B') := \text{Hom}(A, A') \otimes \text{Hom}(B, B')$  and the composition map is defined by  $(f_1 \otimes g_1)(f_2 \otimes g_2) := (-1)^{\deg(g_1)\deg(f_2)} f_1 f_2 \otimes g_1 g_2$ .

Note that the DG categories  $\mathcal{A} \otimes \mathcal{B}$  and  $\mathcal{B} \otimes \mathcal{A}$  are canonically isomorphic. In the above notation, the isomorphism functor  $\phi$  is

$$\phi(A \otimes B) = (B \otimes A), \quad \phi(f \otimes g) = (-1)^{\deg(f)\deg(g)} (g \otimes f). \quad (4.1)$$

# $\mathcal{V}$ -функтори

DEFINITION 3.5.1. A  **$\mathcal{V}$ -functor**  $F: \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$  between  $\mathcal{V}$ -categories is given by an object map  $\underline{\mathcal{C}} \ni x \mapsto Fx \in \underline{\mathcal{D}}$  together with morphisms

$$\underline{\mathcal{C}}(x, y) \xrightarrow{F_{x,y}} \underline{\mathcal{D}}(Fx, Fy)$$

in  $\mathcal{V}$  for each  $x, y \in \underline{\mathcal{C}}$  such that the following diagrams commute for all  $x, y, z \in \underline{\mathcal{C}}$ :

$$\begin{array}{ccc}
 \underline{\mathcal{C}}(y, z) \times \underline{\mathcal{C}}(x, y) & \xrightarrow{\circ} & \underline{\mathcal{C}}(x, z) \\
 \downarrow F_{y,z} \times F_{x,y} & & \downarrow F_{x,z} \\
 \underline{\mathcal{D}}(Fy, Fz) \times \underline{\mathcal{D}}(Fx, Fy) & \xrightarrow{\circ} & \underline{\mathcal{D}}(Fx, Fz)
 \end{array}
 \qquad
 \begin{array}{ccc}
 * & \xrightarrow{\text{id}_x} & \underline{\mathcal{C}}(x, x) \\
 & \searrow \text{id}_{Fx} & \downarrow F_{x,x} \\
 & & \underline{\mathcal{D}}(Fx, Fx)
 \end{array}$$

# dg-еквівалентності

Given a DG category  $\mathcal{A}$ , one defines the graded category  $\text{Ho}^*(\mathcal{A})$  with  $\text{Ob}\text{Ho}^*(\mathcal{A}) = \text{Ob } \mathcal{A}$  by replacing each Hom complex by the direct sum of its cohomology groups. We call  $\text{Ho}^*(\mathcal{A})$  the *graded homotopy category* of  $\mathcal{A}$ . Restricting ourselves to the 0th cohomology of the Hom complexes, we get the *homotopy category*  $\text{Ho}(\mathcal{A})$ .

Two objects  $A, B \in \text{Ob } \mathcal{A}$  are called DG *isomorphic* (or, simply, isomorphic) if there exists an invertible degree-zero morphism  $f \in \text{Hom}(A, B)$ . We say that  $A, B$  are *homotopy-equivalent* if they are isomorphic in  $\text{Ho}(\mathcal{A})$ .

A DG functor between DG categories  $F : \mathcal{A} \rightarrow \mathcal{B}$  is said to be a *quasi-equivalence* if  $\text{Ho}^*(F) : \text{Ho}^*(\mathcal{A}) \rightarrow \text{Ho}^*(\mathcal{B})$  is full and faithful and  $\text{Ho}(F)$  is essentially surjective. We say that  $F$  is a DG *equivalence* if it is full and faithful and every object of  $\mathcal{B}$  is DG isomorphic to an object of  $F(\mathcal{A})$ . Certainly, a DG equivalence is a quasi-equivalence. DG categories  $\mathcal{C}$  and  $\mathcal{D}$  are called *quasi-equivalent* if there exist DG categories  $\mathcal{A}_1, \dots, \mathcal{A}_n$  and a chain of quasi-equivalences

$$\mathcal{C} \leftarrow \mathcal{A}_1 \longrightarrow \cdots \leftarrow \mathcal{A}_n \longrightarrow \mathcal{D}. \quad (4.2)$$

# Еквівалентності

## Definition

A DG functor between DG categories  $F : \mathcal{A} \rightarrow \mathcal{B}$  –  
еквівалентність =  $A_\infty$ -еквівалентність, якщо

- ▶  $\forall X, Y \in \text{Ob } \mathcal{A}$  ланцюгове відображення  
 $F_{X,Y} : \mathcal{A}(X, Y) \rightarrow \mathcal{B}(XF, YF)$  гомотопічно оборотне,  
тобто,  $[F_{X,Y}] \in H^0 \underline{\mathbf{dg}}(\mathcal{A}(X, Y), \mathcal{B}(XF, YF))$  – ізоморфізм,
- ▶  $\text{Ho}(F)$  is essentially surjective on objects, тобто,  $\forall U \in \text{Ob } \mathcal{B}$   
 $\exists V \in \text{Ob } \mathcal{A} \exists$  ізоморфізм  $r \in H^0 \mathcal{B}(U, VF)$ .

## End

The integral, called an end, is the limit of a particular diagram constructed from a functor that is both covariant and contravariant in  $\mathcal{C}$ . Given  $H : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{E}$ , the end  $\int_{\mathcal{C}} H$  is an object of  $\mathcal{E}$  equipped with arrows  $\int_{\mathcal{C}} H \rightarrow H(c, c)$  for each  $c \in \mathcal{C}$  that are collectively universal with the property that the diagram

$$\begin{array}{ccc} \int_{\mathcal{C}} H & \longrightarrow & H(c', c') \\ \downarrow & & \downarrow H(f, c') \\ H(c, c) & \xrightarrow{H(c, f)} & H(c, c') \end{array}$$

commutes for each  $f : c \rightarrow c'$  in  $\mathcal{C}$ .

Equivalently,  $\int_{\mathcal{C}} H$  is the equalizer of the diagram

$$\int_{\mathcal{C}} H \dashrightarrow \prod_{c \in \text{Ob } \mathcal{C}} H(c, c) \rightrightarrows \prod_{f \in \text{Mor } \mathcal{C}} H(\text{dom } f, \text{cod } f).$$

**Theorem 1.4.1.** *Given functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  between small categories we have the canonical isomorphism of sets*

$$\text{Cat}(\mathcal{C}, \mathcal{D})(F, G) \cong \int_C \mathcal{D}(FC, GC). \quad (1.47)$$

*Proof* A wedge  $\tau_C : Y \rightarrow \mathcal{D}(FC, GC)$  consists of a function  $y \mapsto (\tau_{C,y} : FC \rightarrow GC \mid C \in \mathcal{C})$ , which is natural in  $C \in \mathcal{C}$  (this is simply a rephrasing of the wedge condition): the equation

$$G(f) \circ \tau_{C,y} = \tau_{C',y} \circ F(f) \quad (1.48)$$

valid for any  $f : C \rightarrow C'$ , means that for a fixed  $y \in Y$  the arrows  $\tau_C$  form the components of a natural transformation  $F \Rightarrow G$ ; thus, there exists a unique way to close the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tau_C} & \mathcal{D}(FC, GC) \\ & \searrow^h & \uparrow \\ & \text{Cat}(\mathcal{C}, \mathcal{D})(F, G) & \end{array} \quad (1.49)$$

with a function sending  $y \mapsto \tau_{-,y} \in \prod_{C \in \mathcal{C}} \mathcal{D}(FC, GC)$ , and where  $\text{Cat}(\mathcal{C}, \mathcal{D})(F, G) \rightarrow \mathcal{D}(FC, GC)$  is the wedge sending a natural transformation to its  $c$ -component; the diagram commutes for a single  $h : Y \rightarrow \text{Cat}(\mathcal{C}, \mathcal{D})(F, G)$ , and this is precisely the desired universal property for  $\text{Cat}(\mathcal{C}, \mathcal{D})(F, G)$  to be  $\int_C \mathcal{D}(FC, GC)$ .  $\square$

## Перетворення

Given DG categories  $\mathcal{A}$  and  $\mathcal{B}$ , the collection of covariant DG functors  $\mathcal{A} \rightarrow \mathcal{B}$  is itself the collection of objects of a DG category, which we denote by  $\text{Fun}_{\text{DG}}(\mathcal{A}, \mathcal{B})$ . Namely, let  $\phi$  and  $\psi$  be two DG functors. Put  $\text{Hom}^k(\phi, \psi)$  equal to the set of natural transformations  $t : \phi^{\text{gr}} \rightarrow \psi^{\text{gr}}[k]$  of graded functors from  $\mathcal{A}^{\text{gr}}$  to  $\mathcal{B}^{\text{gr}}$ . This means that for any morphism  $f \in \text{Hom}_{\mathcal{A}}^s(A, B)$  one has

$$\psi(f) \cdot t(A) = (-1)^{ks} t(B) \cdot \phi(f). \quad (4.3)$$

On each  $A \in \mathcal{A}$ , the differential of the transformation  $t$  is equal to  $(dt)(A)$  (one easily checks that this is well defined). Thus, the closed transformations of degree zero are the DG transformations of DG functors. A similar definition gives us the DG category consisting of the contravariant DG functors  $\text{Fun}_{\text{DG}}(\mathcal{A}^0, \mathcal{B}) = \text{Fun}_{\text{DG}}(\mathcal{A}, \mathcal{B}^0)$  from  $\mathcal{A}$  to  $\mathcal{B}$ .

$$\begin{array}{ccc} \phi(A) & \xrightarrow{\phi(f)} & \phi(B) \\ t(A) \downarrow & (-1)^{tf} & \downarrow t(B) \\ \psi(A) & \xrightarrow{\psi(f)} & \psi(B) \end{array} \quad \begin{array}{c} t = \{t(A) \mid A \in \text{Ob } \mathcal{A}\} \\ t(A) \in \mathcal{B}(\phi(A), \psi(A))^k \\ \forall f \in \mathcal{A}(A, B)^s \end{array}$$

Modulo size issues, the category  $\mathcal{V}\text{-Cat}$  is closed: given  $\mathcal{V}$ -categories  $\underline{\mathcal{D}}$  and  $\underline{\mathcal{M}}$  where  $\underline{\mathcal{D}}$  is small, define a  $\mathcal{V}$ -category  $\underline{\mathcal{M}}^{\underline{\mathcal{D}}}$  whose objects are  $\mathcal{V}$ -functors  $F, G: \underline{\mathcal{D}} \rightrightarrows \underline{\mathcal{M}}$  and whose hom-objects, taking inspiration from [1.2.8] are defined by the formula

$$(7.3.2) \quad \underline{\mathcal{M}}^{\underline{\mathcal{D}}}(F, G) = \int_{d \in \underline{\mathcal{D}}} \underline{\mathcal{M}}(Fd, Gd).$$

the formula for the enriched end defining the hom-object of  $\mathcal{V}$ -natural transformations is the equalizer

$$(7.3.3) \quad \int_{\underline{\mathcal{D}}} \underline{\mathcal{M}}(Fd, Gd) := \text{eq} \left( \prod_d \underline{\mathcal{M}}(Fd, Gd) \rightrightarrows \prod_{d,d'} \underline{\mathcal{V}}(\underline{\mathcal{D}}(d, d'), \underline{\mathcal{M}}(Fd, Gd')) \right).$$

The component of the top arrow indexed by the ordered pair  $d, d'$  projects to  $\underline{\mathcal{M}}(Fd, Gd)$  and composes in the second coordinate with  $\underline{\mathcal{D}}(d, d')$ . The analogous component of the bottom arrow projects to  $\underline{\mathcal{M}}(Fd', Gd')$  and precomposes in the first coordinate with  $\underline{\mathcal{D}}(d, d')$ .

Верхня стрілка в (7.3.3) – це

$$\prod_d \underline{\mathcal{M}}(Fd, Gd) \xrightarrow{\text{pr}_d} \underline{\mathcal{M}}(Fd, Gd) \rightarrow \underline{\mathcal{V}}^I(\underline{\mathcal{D}}(d, d'), \underline{\mathcal{M}}(Fd, Gd')),$$

де останній морфізм приходить з парування

$$\underline{\mathcal{M}}(Fd, Gd) \otimes \underline{\mathcal{D}}(d, d') \xrightarrow{1 \otimes G} \underline{\mathcal{M}}(Fd, Gd) \otimes \underline{\mathcal{M}}(Gd, Gd') \xrightarrow{\cdot} \underline{\mathcal{M}}(Fd, Gd').$$

Нижня стрілка в (7.3.3) – це

$$\prod_d \underline{\mathcal{M}}(Fd, Gd) \xrightarrow{\text{pr}_{d'}} \underline{\mathcal{M}}(Fd', Gd') \rightarrow \underline{\mathcal{V}}^I(\underline{\mathcal{D}}(d, d'), \underline{\mathcal{M}}(Fd, Gd')),$$

де останній морфізм приходить з композиції

$$\begin{array}{ccc}
 \underline{\mathcal{M}}(Fd', Gd') \otimes \underline{\mathcal{D}}(d, d') & \longrightarrow & \underline{\mathcal{M}}(Fd, Gd') \\
 c \downarrow & \nearrow & \uparrow \\
 \underline{\mathcal{D}}(d, d') \otimes \underline{\mathcal{M}}(Fd', Gd') & \xrightarrow{F \otimes 1} & \underline{\mathcal{M}}(Fd, Fd') \otimes \underline{\mathcal{M}}(Fd', Gd')
 \end{array}$$

тоді як діагональній стрілці відповідало б

$$\underline{\mathcal{M}}(Fd', Gd') \rightarrow \underline{\mathcal{V}}^r(\underline{\mathcal{D}}(d, d'), \underline{\mathcal{M}}(Fd, Gd')).$$

# **dg**-модулі над **dg**-категорією

We denote the DG category  $\text{Fun}_{\text{DG}}(\mathcal{B}, \text{DG}(k))$  by  $\mathcal{B}\text{-mod}$  and call it the category of DG  $\mathcal{B}$ -modules. There is a natural covariant DG functor  $h : \mathcal{A} \rightarrow \mathcal{A}^0\text{-mod}$  (the Yoneda embedding) defined by  $h^A(B) := \text{Hom}_{\mathcal{A}}(B, A)$ . As in the “classical” case one verifies that the functor  $h$  is full and faithful, that is,

$$\text{Hom}_{\mathcal{A}}(A, A') = \text{Hom}_{\mathcal{A}^0\text{-mod}}(h^A, h^{A'}). \quad (4.4)$$

Moreover, for any  $F \in \mathcal{A}^0\text{-mod}$ ,  $A \in \mathcal{A}$ ,

$$\text{Hom}_{\mathcal{A}^0\text{-mod}}(h^A, F) = F(A). \quad (4.5)$$

The  $\mathcal{A}^0$ -DG-modules  $h^A$ ,  $A \in \mathcal{A}$ , are called *free*. An  $\mathcal{A}^0$ -DG-module  $F$  is called *semifree* if it has a filtration

$$0 = F_0 \subset F_1 \subset \cdots = F \quad (4.6)$$

such that  $F_{i+1}/F_i$  is isomorphic to a direct sum of shifted free  $\mathcal{A}^0$ -DG-modules  $h^A[n]$ ,  $n \in \mathbb{Z}$ . The full subcategory of semifree  $\mathcal{A}^0$ -DG-modules is denoted by  $\text{SF}(\mathcal{A})$ .

An  $\mathcal{A}^0$ -DG-module  $F$  is called acyclic if the complex  $F(A)$  is acyclic for all  $A \in \mathcal{A}$ . Let  $D(\mathcal{A})$  denote the *derived category* of  $\mathcal{A}^0$ -DG-modules, that is,  $D(\mathcal{A})$  is the Verdier quotient of the homotopy category  $\text{Ho}(\mathcal{A}^0\text{-mod})$  by the subcategory of acyclic DG-modules.

## $\mathbf{dg}$ -категорії з формальним зсувом

Given a DG category  $\mathcal{A}$ , one can associate to it a triangulated category  $\mathcal{A}^{\text{tr}}$  [5]. It is defined as the homotopy category of a certain DG category  $\mathcal{A}^{\text{pre-tr}}$ . The idea of the definition of  $\mathcal{A}^{\text{pre-tr}}$  is to formally add cones of all morphisms, cones of morphisms between cones, and so forth.

First, we need to clarify the notion of a “formal shift” of an object.

**Definition 4.5.** Define the DG category  $\bar{\mathcal{A}}$  as follows:

$$\text{Ob } \bar{\mathcal{A}} = \{A[n] \mid A \in \text{Ob } \mathcal{A}, n \in \mathbb{Z}\}, \quad (4.12)$$

and define

$$\text{Hom}_{\bar{\mathcal{A}}} (A[k], B[n]) = \text{Hom}_{\mathcal{A}} (A, B)[n - k] \quad (4.13)$$

as *graded vector spaces*. If  $f \in \text{Hom}_{\mathcal{A}} (A, B)$  is considered as an element of  $\text{Hom}_{\bar{\mathcal{A}}} (A[k], B[n])$  under the above identification, then the differentials are related by the formula

$$d_{\bar{\mathcal{A}}} (f) = (-1)^n d_{\mathcal{A}} (f). \quad (4.14)$$

Notice, for example, that the differential in  $\text{Hom}_{\bar{\mathcal{A}}} (A[1], B[1])$  is equal to *minus* the differential in  $\text{Hom}_{\mathcal{A}} (A, B)$ .

## Алгебра в категорії $\mathbf{dg}$ -категорій

The category of differential graded categories  $\mathbf{dg}\text{-}\mathcal{C}at$  equipped with the tensor product  $\boxtimes$  is a symmetric monoidal category. Let us study an algebra in this category, which will be used to define the functor of shifts. Let  $\mathcal{Z}$  be a differential graded quiver with  $\mathbf{Ob}\mathcal{Z} = \mathbb{Z}$ ,  $\mathcal{Z}(m, n) = \mathbb{k}[n - m]$  and zero differential. Consider an arbitrary  $\mathbb{k}$ -linear graded category structure of  $\mathcal{Z}$  given by isomorphisms

$$\begin{aligned}\mu' &= \phi'(l, m, n) : \mathcal{Z}(l, m) \otimes_{\mathbb{k}} \mathcal{Z}(m, n) \\ &= \mathbb{k}[m - l] \otimes \mathbb{k}[n - m] \rightarrow \mathbb{k}[n - l] = \mathcal{Z}(l, n), \\ 1s^{m-l} \otimes 1s^{n-m} &\mapsto \phi'(l, m, n)s^{n-l}.\end{aligned}$$

It is specified by a function  $\phi' : \mathbb{Z}^3 \rightarrow \mathbb{k}^\times$  with values in the multiplicative group  $\mathbb{k}^\times$ . The associativity of composition

$$(1 \otimes \mu')\mu' = (\mu' \otimes 1)\mu' : \mathcal{Z}(k, l) \otimes \mathcal{Z}(l, m) \otimes \mathcal{Z}(m, n) \rightarrow \mathcal{Z}(k, n)$$

implies that  $\phi'$  is a cycle:

$$\phi'(l, m, n) \cdot \phi'(k, l, n) = \phi'(k, l, m) \cdot \phi'(k, m, n).$$

An arbitrary such cycle  $\phi'$  is a boundary

$$\phi'(l, m, n) = \xi(l, m) \cdot \xi(m, n) \cdot \xi(l, n)^{-1} \quad (1)$$

of a function  $\xi : \mathbb{Z}^2 \rightarrow \mathbb{k}^\times$ , namely,  $\xi(l, m) = \phi'(0, l, m)$ .

The graded quiver automorphism  $\xi : \mathcal{L} \rightarrow \mathcal{L}$ ,  $n \mapsto n$ ,  $\xi(m, n) : \mathcal{L}(m, n) \rightarrow \mathcal{L}(m, n)$  equips  $\mathcal{L}$  with another (isomorphic) category structure  $(\mathcal{L}, \mu)$  with  $\phi(l, m, n) = 1$ , so that  $\xi : (\mathcal{L}, \mu') \rightarrow (\mathcal{L}, \mu)$  is a category isomorphism.

In the following we shall consider only composition  $\mu$  in  $\mathcal{L}$  specified by the function  $\phi(l, m, n) = 1$ . The elements

$1 \in \mathbb{k} = \mathcal{L}(n, n)$  are identity morphisms of  $\mathcal{L}$ .

We equip the object  $\mathcal{L}$  of  $(\mathbf{dg}\text{-}\mathcal{C}at, \boxtimes)$  with an algebra structure, given by multiplication – differential graded functor

$$\otimes_\psi : \mathcal{L} \boxtimes \mathcal{L} \rightarrow \mathcal{L}, \quad m \times n \mapsto m + n,$$

$$\begin{aligned} \otimes_\psi = \psi(n, m, k, l) &: (\mathcal{L} \boxtimes \mathcal{L})(n \times m, k \times l) \\ &= \mathcal{L}(n, k) \otimes \mathcal{L}(m, l) \rightarrow \mathcal{L}(n + m, k + l), \\ 1s^{k-n} \otimes 1s^{l-m} &\mapsto \psi(n, m, k, l)s^{k+l-n-m}. \end{aligned}$$

We assume that the function  $\psi : \mathbb{Z}^4 \rightarrow \mathbb{k}$  takes values in  $\mathbb{k}^\times$ .

Being a functor,  $\otimes_\psi$  has to satisfy the equation:

$$\psi(a, b, c, d) \cdot \psi(c, d, e, f) = (-1)^{(d-b)(e-c)} \psi(a, b, e, f).$$

It specifies the boundary of the 2-cochain  $\psi : \mathbb{Z}^4 \rightarrow \mathbb{k}^\times$ . Generic solution to this equation is

$$\psi(a, b, c, d) = (-1)^{c(b-d)} \chi(a, b) \cdot \chi(c, d)^{-1} \quad (2)$$

for some function  $\chi : \mathbb{Z}^2 \rightarrow \mathbb{k}^\times$  (take  $\chi(a, b) = \psi(a, b, 0, 0)$ ).

Associativity of the algebra  $(\mathcal{L}, \otimes_\psi)$

$$\begin{array}{ccc}
 (\mathcal{L} \boxtimes \mathcal{L} \boxtimes \mathcal{L})(a \times b \times c, d \times e \times f) & \xrightarrow{1 \boxtimes \otimes_\psi} & (\mathcal{L} \boxtimes \mathcal{L})(a \times (b+c), d \times (e+f)) \\
 \downarrow \otimes_\psi \boxtimes 1 & = & \downarrow \otimes_\psi \\
 (\mathcal{L} \boxtimes \mathcal{L})((a+b) \times c, (d+e) \times f) & \xrightarrow{\otimes_\psi} & \mathcal{L}(a+b+c, d+e+f)
 \end{array}$$

is expressed by the equation

$$\psi(a, b, d, e) \cdot \psi(a+b, c, d+e, f) = \psi(b, c, e, f) \cdot \psi(a, b+c, d, e+f).$$

It means that the function  $\overline{\psi} : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{k}^\times$ ,  
 $\overline{\psi}(a, b, c, d) = \psi(a, c, b, d)$  is a 2-cocycle on the group  $\mathbb{Z}^2$ . For a  
 graded category  $\mathcal{L}$  with multiplication (1) related to  $(\mathcal{L}, \mu)$  via  
 the category isomorphism  $\xi : (\mathcal{L}, \mu') \rightarrow (\mathcal{L}, \mu)$ ,  $n \mapsto n$ ,  
 $\xi(m, n) : \mathcal{L}(m, n) \rightarrow \mathcal{L}(m, n)$ , we would get the 2-cocycle

$$\overline{\psi}'(a, b, c, d) = \xi(a, b) \cdot \xi(c, d) \cdot \overline{\psi}(a, b, c, d) \cdot \xi(a + c, b + d)^{-1},$$

cohomologous to  $\overline{\psi}$ .

Plugging (2) into equation for  $\psi$ , we reduce the latter to

$$\begin{aligned} & \chi(a, b) \cdot \chi(a, b + c)^{-1} \cdot \chi(a + b, c) \cdot \chi(b, c)^{-1} \\ &= \chi(d, e) \cdot \chi(d, e + f)^{-1} \cdot \chi(d + e, f) \cdot \chi(e, f)^{-1}. \end{aligned}$$

The common value of the left and right hand sides does not  
 depend on the arguments  $a, b, c, d, e, f \in \mathbb{Z}$ . Setting  $d = e = f = 0$   
 we find that this constant equals 1. Thus,  $\chi : \mathbb{Z}^2 \rightarrow \mathbb{k}^\times$  is a  
 2-cocycle on the group  $\mathbb{Z}^2$  with values in  $\mathbb{k}^\times$ . Since  $\mathbb{Z}^2$  is a free  
 group, its cohomology group  $H^2(\mathbb{Z}^2, \mathbb{k}^\times)$  vanishes.

Therefore, the cocycle  $\chi$  has the form

$$\chi(a, b) = \lambda(a) \cdot \lambda(b) \cdot \lambda(a+b)^{-1}$$

for some function  $\lambda : \mathbb{Z} \rightarrow \mathbb{k}^\times$ . The corresponding function  $\psi : \mathbb{Z}^4 \rightarrow \mathbb{k}^\times$  is given by the formula

$$\begin{aligned} & \psi_\lambda(a, b, c, d) \\ &= (-1)^{c(b-d)} \lambda(a) \cdot \lambda(b) \cdot \lambda(a+b)^{-1} \cdot \lambda(c)^{-1} \cdot \lambda(d)^{-1} \cdot \lambda(c+d). \end{aligned}$$

Denote the multiplication functor  $\otimes_{\psi_\lambda}$  also by  $\otimes^\lambda$ . Define an automorphism of the graded category  $\mathcal{L}$  as  $\bar{\lambda} : \mathcal{L} \rightarrow \mathcal{L}$ ,  $n \mapsto n$ ,  $\bar{\lambda} = \lambda(m)^{-1} \cdot \lambda(n) : \mathcal{L}(m, n) \rightarrow \mathcal{L}(m, n)$ .  $\bar{\lambda} : (\mathcal{L}, \otimes^1) \rightarrow (\mathcal{L}, \otimes^\lambda)$  is an algebra isomorphism, where the first algebra uses

$$\lambda_1(a) = 1, \quad \psi_1(a, b, c, d) = (-1)^{c(b-d)}.$$

The multiplicative cocycle  $\bar{\psi}_1 \in Z^2(\mathbb{Z}^2, \mathbb{k}^\times)$  comes via the homomorphism  $\mathbb{Z}/2 \rightarrow \mathbb{k}^\times$ ,  $a \mapsto (-1)^a$ , from the additive cocycle  $\mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{Z}/2$ ,  $(a, c, b, d) \mapsto c(b-d) \pmod{2}$ , which represents the only non-trivial element of the cohomology group  $H^2(\mathbb{Z}^2, \mathbb{Z}/2) = \mathbb{Z}/2$  with values in the trivial  $\mathbb{Z}^2$ -module  $\mathbb{Z}/2$ .

In the following we shall consider only multiplication  $\otimes_{\mathcal{L}} = \otimes^1$  in  $\mathcal{L}$  specified by the function  $\psi_1(a, b, c, d) = (-1)^{c(b-d)}$ .

Clearly, the algebra  $(\mathcal{L}, \otimes_{\mathcal{L}})$  is unital with the unit

$$\eta_{\mathcal{L}} : \mathbb{1} \rightarrow \mathcal{L}, * \mapsto 0, \text{id} : \mathbb{1}(*, *) = \mathbb{k} \rightarrow \mathcal{L}(0, 0).$$

## Функтор зсувів

Given a **dg**-category  $\mathcal{A}$ , we produce another one  $\mathcal{A}^{[l]} = \mathcal{A} \boxtimes \mathcal{Z}$ , obtained by adding formal shifts of objects. The set of objects is

$$\text{Ob } \mathcal{A}^{[l]} = \{X[n] = (X, n) \mid X \in \text{Ob } \mathcal{A}, n \in \mathbb{Z}\} = \text{Ob } \mathcal{A} \times \mathbb{Z}.$$

The graded  $\mathbb{k}$ -module of morphisms is

$\mathcal{A}^{[l]}(X[n], Y[m]) = \mathcal{A}(X, Y) \otimes \mathbb{k}[m-n]$ . We identify it with  $\mathcal{A}(X, Y)[m-n]$ . Given a morphism of **dg**-categories  $f : \mathcal{A} \rightarrow \mathcal{B}$ , we define another one  $f^{[l]} = f \boxtimes 1_{\mathcal{Z}} : \mathcal{A}^{[l]} \rightarrow \mathcal{B}^{[l]}$ . On objects it acts as  $\text{Ob } f^{[l]} : X[n] \mapsto (Xf)[n]$ . Commutative diagram

$$\begin{array}{ccccc} \mathcal{A}(X, Y)[m-n] & \xrightarrow{s^{n-m}} & \mathcal{A}(X, Y) \otimes \mathbb{k} & \xrightarrow{1 \otimes s^{m-n}} & \mathcal{A}(X, Y) \otimes \mathcal{Z}(n, m) \\ f^{[m-n]} \downarrow & & \downarrow f \otimes 1 & & \downarrow f \otimes 1 \\ \mathcal{B}^{[l]}(Xf, Yf)[m-n] & \xrightarrow{s^{n-m}} & \mathcal{B}(Xf, Yf) \otimes \mathbb{k} & \xrightarrow{1 \otimes s^{m-n}} & \mathcal{B}(Xf, Yf) \otimes \mathcal{Z}(n, m) \end{array}$$

describes the action of  $-^{[l]}$  on morphisms in another presentation:

$$f^{[m-n]} = s^{n-m} f s^{m-n} : \mathcal{A}(X, Y)[m-n] \rightarrow \mathcal{B}(Xf, Yf)[m-n].$$

## **dg**-категорії замкнені відносно зсувів

Диференціал на  $\mathcal{A}^{[]} \mathbb{I}$  обчислюється як

$$\begin{aligned} d^{[]} = (-1)^{m-n} s^{n-m} ds^{m-n} : \mathcal{A}^{[]}(\mathcal{X}[n], \mathcal{Y}[m]) &= \mathcal{A}(\mathcal{X}, \mathcal{Y})[m-n] \\ &\rightarrow \mathcal{A}(\mathcal{X}, \mathcal{Y})[m-n] = \mathcal{A}^{[]}(\mathcal{X}[n], \mathcal{Y}[m]). \end{aligned}$$

Композиція для  $\mathcal{A}^{[]} \mathbb{I}$  обчислюється як

$$\begin{aligned} m^{[]} = (s^{p-n} \otimes s^{k-p})^{-1} ms^{k-n} : \mathcal{A}^{[]}(\mathcal{X}[n], \mathcal{Y}[p]) \otimes \mathcal{A}^{[]}(\mathcal{Y}[p], \mathcal{Z}[k]) &= \\ \mathcal{A}(\mathcal{X}, \mathcal{Y})[p-n] \otimes \mathcal{A}(\mathcal{Y}, \mathcal{Z})[k-p] &\rightarrow \mathcal{A}(\mathcal{X}, \mathcal{Z})[k-n] = \mathcal{A}^{[]}(\mathcal{X}[n], \mathcal{Z}[k]). \end{aligned}$$

### Definition

We say that a **dg**-category  $\mathcal{C}$  is **closed under shifts** if every object  $\mathcal{X}[n]$  of  $\mathcal{C}^{[]} \mathbb{I}$  is homotopy isomorphic in  $\mathcal{C}^{[]} \mathbb{I}$  to some object  $\mathcal{Y}[0]$ ,  $\mathcal{Y} = [\mathcal{X}, n] \in \text{Ob } \mathcal{C}$ .

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