2. Триангульовані категорії. Навколо похідних категорій

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Властивості $\mathsf{K}(\mathscr{C})$ як триангульованої категорії

Proposition 1.4.4. *The collection of distinguished triangles in* $\mathsf{K}(\mathscr{C})$ *satisfies the following properties,* (TR 0)–(TR 5).

(TR 0) *A triangle isomorphic to a distinguished triangle is distinguished.*

(TR 1) *For any* $X \in \mathrm{Ob}(\mathsf{K}(\mathscr{C}))$, $X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$ *is a distinguished triangle.*

(TR 2) *Any* $f : X \to Y$ *in* $\mathsf{K}(\mathscr{C})$ *can be embedded in a distinguished triangle* $X \xrightarrow{f} Y \to Z \to X[1]$.

(TR 3) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *is a distinguished triangle if and only if* $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ *is a distinguished triangle.*

(TR 4) *Given two distinguished triangles* $X \xrightarrow{f} Y \to Z \to X[1]$ *and* $X' \xrightarrow{f'} Y' \to Z' \to X'[1]$, *a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

can be embedded in a morphism of triangles (not necessarily unique).

(TR 5) (octahedral axiom). *Suppose given distinguished triangles:*

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow X[1]\ ,$$

$$Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow Y[1]\ ,$$

$$X \xrightarrow{g \circ f} Z \longrightarrow Y' \longrightarrow X[1]\ ,$$

then there exists a distinguished triangle

$$Z' \to Y' \to X' \to Z'[1]$$

such that the following diagram is commutative:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & X[1] \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & \big\downarrow{\scriptstyle g} & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{X[1]}} \\
X & \xrightarrow{g \circ f} & Z & \longrightarrow & Y' & \longrightarrow & X[1] \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_Z} & & \big\downarrow & & \big\downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & Y[1] \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{X'}} & & \big\downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z'[1]\ .
\end{array}$$

Доведення властивостей $\mathbf{K}(\mathscr{C})$ як трианг. категорії

Proof. The properties (TR 0) and (TR 2) are obvious, and (TR 3) follows from Lemma 1.4.2.

Since the mapping cone of $f : 0 \to X$ is X, the triangle $0 \longrightarrow X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0[1]$ is distinguished. Applying (TR 3) we get (TR 1). Let us prove (TR 4). We may assume that $X \xrightarrow{f} Y \to Z \to X[1]$ and $X' \xrightarrow{f'} Y' \to Z' \to X'[1]$ are $X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{\alpha(f')} M(f') \xrightarrow{\beta(f')} X'[1]$, respectively. We shall construct a morphism $w : M(f) \to M(f')$ such that:

$$(1.4.4) \qquad \begin{cases} w \circ \alpha(f) = \alpha(f') \circ v \ , \\ u[1] \circ \beta(f) = \beta(f') \circ w \ . \end{cases}$$

By the definition of $\mathbf{K}(\mathscr{C})$, there exists $s^n : X^n \to Y'^{n-1}$ such that $v^n \circ f^n - f'^n \circ u^n = s^{n+1} \circ d_X^n + d_{Y'}^{n-1} \circ s^n$. We define $w^n : M(f)^n = X^{n+1} \oplus Y^n \to M(f')^n = X'^{n+1} \oplus Y'^n$ by:

$$w^n = \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix} .$$

Then a direct calculation shows that w is a morphism of complexes and satisfies (1.4.4).

$$w = \begin{pmatrix} u[1] & 0 \\ s \circ \sigma^{-1} & v \end{pmatrix}.$$

Let us prove (TR 5). We may assume $Z' = M(f)$, $X' = M(g)$ and $Y' = M(g \circ f)$. Let us define $u : Z' \to Y'$ and $v : Y' \to X'$ by:

$$u^n : X^{n+1} \oplus Y^n \to X^{n+1} \oplus Z^n \ , \qquad u = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \\ 0 & g^n \end{pmatrix} ,$$

$$v^n : X^{n+1} \oplus Z^n \to Y^{n+1} \oplus Z^n \ , \qquad v^n = \begin{pmatrix} f^{n+1} & 0 \\ 0 & \mathrm{id}_{Z^n} \end{pmatrix} .$$

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{\alpha(f)} & Z' & \xrightarrow{\beta(f)} & X[1] \\
\downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{id}_{X[1]}} \\
X & \xrightarrow{g \circ f} & Z & \xrightarrow{\alpha(g \circ f)} & Y' & \xrightarrow{\beta(g \circ f)} & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_Z} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & Z & \xrightarrow{\alpha(g)} & X' & \xrightarrow{\beta(g)} & Y[1] \\
\downarrow{\scriptstyle \alpha(f)} & & \downarrow{\scriptstyle \alpha(g \circ f)} & & \downarrow{\scriptstyle \mathrm{id}_{X'}} & & \downarrow{\scriptstyle \alpha(f)[1]} \\
Z' & \xrightarrow[u]{} & Y' & \xrightarrow[v]{} & X' & \longrightarrow & Z'[1] .
\end{array}$$

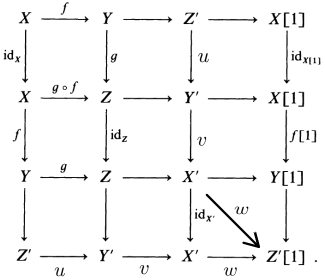
We define $w : X' \to Z'[1]$ as the composite $X' \to Y[1] \to Z'[1]$. Then the diagram in (TR 5) is commutative, and it is enough to show that $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} Z'[1]$ is a distinguished triangle. For that purpose we shall construct an isomorphism $\phi : M(u) \to X'$ and its inverse $\psi : X' \to M(u)$ such that $\phi \circ \alpha(u) = v$ and $\beta(u) \circ \psi = w$. We have:

$$M(u)^n = M(f)^{n+1} \oplus M(g \circ f)^n = X^{n+2} \oplus Y^{n+1} \oplus X^{n+1} \oplus Z^n$$

and $X'^n = M(g)^n = Y^{n+1} \oplus Z^n$. We define ϕ and ψ by:

$$\phi^n = \begin{pmatrix} 0 & \mathrm{id}_{Y^{n+1}} & f^{n+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z^n} \end{pmatrix}, \qquad \psi^n = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{X^{n+1}} \end{pmatrix}.$$

Then one checks easily that ϕ and ψ are morphisms of complexes and $\phi \circ \alpha(u) = v$, $\beta(u) \circ \psi = w$.



Октаедр

We have $\phi \circ \psi = \mathrm{id}_{X'}$. If we define:

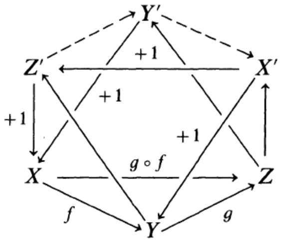
$$s^n : M(u)^n \to M(u)^{n-1} \quad , \qquad s^n = \begin{pmatrix} 0 & 0 & \mathrm{id}_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then:

$$(\mathrm{id}_{M(u)} - \psi \circ \phi)^n = s^{n+1} \circ \mathrm{d}^n_{M(u)} + \mathrm{d}^{n-1}_{M(u)} \circ s^n \ .$$

Hence $\psi \circ \phi$ equals $\mathrm{id}_{M(u)}$ in $\mathbf{K}(\mathscr{C})$. □

Remark 1.4.5. Property (TR 5) may be visualized by the following octahedral diagram:



Триангульована категорія

Let $\mathscr{C}$ be an additive category, together with an automorphism $T : \mathscr{C} \to \mathscr{C}$. We write sometimes [1] for T and $[k]$ for T^k, (i.e. $X[1]$ for $T(X)$, or $f[1]$ for $T(f)$).

A triangle in $\mathscr{C}$ is a sequence of morphisms

$$X \to Y \to Z \to T(X) \ .$$

Definition 1.5.1. *A triangulated category $\mathscr{C}$ consists of the following data and rules.*

(1.5.1) *An additive category $\mathscr{C}$ together with an automorphism $T : \mathscr{C} \to \mathscr{C}$,*

(1.5.2) *a family of triangles, called distinguished triangles.*

These data satisfy the axioms (TR 0)–(TR 5) *of Proposition* 1.4.4 *when setting* $X[1] = T(X)$.

Let $(\mathscr{C}, T)$ and $(\mathscr{C}', T')$ be two triangulated categories. We say that an additive functor F from $\mathscr{C}$ to $\mathscr{C}'$ is a functor of triangulated categories if $F \circ T \simeq T' \circ F$, and F sends distinguished triangles of $\mathscr{C}$ into distinguished triangles of $\mathscr{C}'$.

Clearly, for an additive category $\mathscr{C}$, $\mathbf{K}(\mathscr{C})$ is a triangulated category.

Когомологічний функтор

$\mathscr{C}$ – триангульована категорія, $\mathscr{A} = \Bbbk$-mod.

Definition 1.5.2. *An additive functor $F : \mathscr{C} \to \mathscr{A}$ is called a cohomological functor if for any distinguished triangle $X \to Y \to Z \to T(X)$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact.*

For a cohomological functor F, we write F^k for $F \circ T^k$. Then for any distinguished triangle $X \to Y \to Z \to T(X)$ we obtain a long exact sequence:

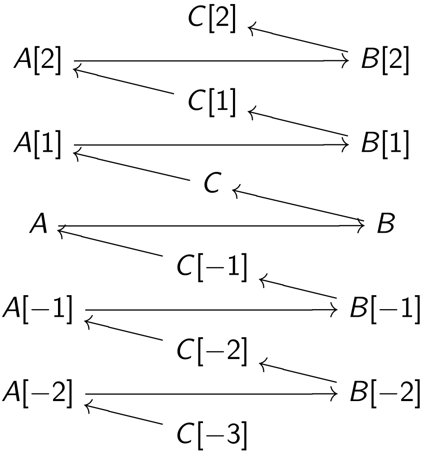
$$\cdots \to F^{k-1}(Z) \to F^k(X) \to F^k(Y) \to F^k(Z) \to F^{k+1}(X) \to \cdots \quad . \tag{1.5.3}$$

Трикутник $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ - скорочений запис послідовності

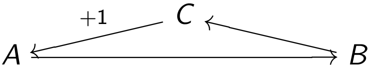
$$\ldots \xrightarrow{g[-1]} C[-1] \xrightarrow{h[-1]} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1] \xrightarrow{g[1]} C[1] \xrightarrow{h[1]} \ldots$$

яку можна візуалізувати як спіраль

Трикутник



що проектується на трикутник



Proposition 1.5.3. (i) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$ *is a distinguished triangle, then* $g \circ f = 0$.

(ii) *For any* $W \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}_{\mathscr{C}}(W, \cdot)$ *and* $\mathrm{Hom}_{\mathscr{C}}(\cdot, W)$ *are cohomological functors.*

Proof. (i) By (TR 1), $X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow T(X)$ is a distinguished triangle. Therefore by (TR 4) there is a morphism $\phi : 0 \to Z$ which makes the following diagram commutative:

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & T(X) \\ \Big\downarrow{\scriptstyle \mathrm{id}_X} & & \Big\downarrow{\scriptstyle f} & & \vdots{\scriptstyle \phi} & & \Big\downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & T(X) . \end{array}$$

Hence $g \circ f = \phi \circ 0 = 0$.

(ii) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$ be a distinguished triangle. In order to show that $\mathrm{Hom}_{\mathscr{C}}(W, \cdot)$ is a cohomological functor, it is enough to show that, for any $\phi \in \mathrm{Hom}_{\mathscr{C}}(W, Y)$ with $g \circ \phi = 0$, we can find $\psi \in \mathrm{Hom}_{\mathscr{C}}(W, X)$, with $\phi = f \circ \psi$. This follows from (TR 1), (TR 3) and (TR 4) which imply that the dotted arrow below can be completed:

$$\begin{array}{ccccccc} W & \xrightarrow{\mathrm{id}_W} & W & \longrightarrow & 0 & \longrightarrow & T(W) \\ \vdots{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow & & \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & T(X) . \end{array}$$

The proof that $\mathrm{Hom}_{\mathscr{C}}(\cdot, W)$ is a cohomological functor is similar. $\square$

5 ізоморфізмів

Corollary 1.5.5. *Let*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle T(\phi)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

be a morphism of distinguished triangles. If ϕ and ψ are isomorphisms, then so is θ.

Proof. For any $W \in \mathrm{Ob}(\mathscr{C})$, let us apply the functor $\mathrm{Hom}_{\mathscr{C}}(W, \cdot)$ to the above diagram. We obtain a commutative diagram whose rows are exact. Since $\mathrm{Hom}_{\mathscr{C}}(W, \phi)$ and $\mathrm{Hom}_{\mathscr{C}}(W, \psi)$ are isomorphisms, as well as $\mathrm{Hom}_{\mathscr{C}}(W, T(\phi))$ and $\mathrm{Hom}_{\mathscr{C}}(W, T(\psi))$, we obtain that $\mathrm{Hom}_{\mathscr{C}}(W, \theta)$ is an isomorphism by Exercise I.8. (5-лема). За лемою Йонеда $\boldsymbol{\theta}$ - ізоморфізм.

$\mathscr{C} = \Bbbk\text{-mod}$.

Proposition 1.5.6. *Let $\mathscr{C}$ be an abelian category. Then the functor $H^0(\cdot) : \mathbf{K}(\mathscr{C}) \to \mathscr{C}$ is a cohomological functor.*

Proof. It is enough to show that if $f : X \to Y$ is a morphism in $\mathbf{C}(\mathscr{C})$, then the sequence

$$H^0(Y) \to H^0(M(f)) \to H^0(X[1])$$

is exact.

Since $0 \to Y \to M(f) \to X[1] \to 0$ is an exact sequence in $\mathbf{C}(\mathscr{C})$, the result follows from Proposition 1.3.6. □

Definition 1.5.7. *Let $\mathscr{C}$ be an abelian category and let $f : X \to Y$ be a morphism in $\mathbf{K}(\mathscr{C})$. One says that f is a quasi-isomorphism (qis for short) if $H^n(f)$ is an isomorphism for each n.*

Hence f is a *qis* if and only if $H^n(M(f)) = 0$ for each n. If f is a *qis*, one writes $X \xrightarrow[qis]{} Y$, for short.

Notations 1.5.8. Let $\mathscr{C}$ be a triangulated category. In the subsequent sections we shall often write $X \longrightarrow Y \longrightarrow Z \xrightarrow[+1]{}$ instead of $X \to Y \to Z \to T(X)$, to denote a distinguished triangle.

Let $\mathscr{C}$ be a category, and let S be a family of morphisms in $\mathscr{C}$.

Definition 1.6.1. *One says that S is a multiplicative system if it satisfies* (S 1)–(S 4) *below.*

(S 1) *For any $X \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{id}_X \in S$.*
(S2) *For any pair (f, g) of S such that the composition $g \circ f$ exists, $g \circ f \in S$.*
(S 3) *Any diagram:*

$$\begin{array}{ccc} & & Z \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with $g \in S$, may be completed to a commutative diagram:

$$\begin{array}{ccc} W & \longrightarrow & Z \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with $h \in S$. Ditto with all the arrows reversed.
(S4) *If f and g belong to $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, the following conditions are equivalent:*
(i) *there exists $t : Y \to Y'$, $t \in S$, such that $t \circ f = t \circ g$,*
(ii) *there exists $s : X' \to X$, $s \in S$, such that $f \circ s = g \circ s$.*

Локалізація

Definition 1.6.2. *Let $\mathscr{C}$ be a category, S a multiplicative system. The category $\mathscr{C}_S$, called the localization of $\mathscr{C}$ by S, is defined by:*

(1.6.1) $$\mathrm{Ob}(\mathscr{C}_S) = \mathrm{Ob}(\mathscr{C}) \ ,$$

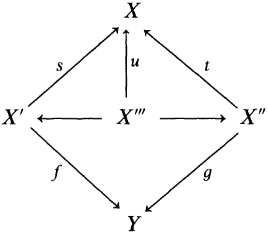
(1.6.2) *for any pair (X, Y) of $\mathrm{Ob}(\mathscr{C})$,*

$$\mathrm{Hom}_{\mathscr{C}_S}(X, Y) = \{(X', s, f); X' \in \mathrm{Ob}(\mathscr{C}), s : X' \to X, f : X' \to Y, s \in S\}/\mathscr{R}$$

where $\mathscr{R}$ is the following equivalence relation:

$$(X', s, f)\mathscr{R}(X'', t, g)$$

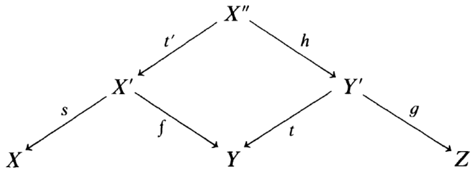
iff there exists a commutative diagram



with $u \in S$.

Композиція в локалізації

The composition of $(X', s, f) \in \mathrm{Hom}_{\mathscr{C}_S}(X, Y)$ *and* $(Y', t, g) \in \mathrm{Hom}_{\mathscr{C}_S}(Y, Z)$ *is defined as follows. We use* (S 3) *to find a commutative diagram*:



with $t' \in S$, *and we set*:

$$(Y', t, g) \circ (X', s, f) = (X'', s \circ t', g \circ h) \ .$$

One sees easily, using the axioms (S 1)–(S 4), that $\mathscr{C}_S$ is a category.

We shall denote by Q the functor:

$$Q : \mathscr{C} \to \mathscr{C}_S$$

defined by $Q(X) = X$ for $X \in \mathrm{Ob}(\mathscr{C})$, and $Q(f) = (X, \mathrm{id}_X, f)$ for $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Proposition 1.6.3. (i) *For* $s \in S$, $Q(s)$ *is an isomorphism in* $\mathscr{C}_S$.

(ii) *Let* $\mathscr{C}'$ *be another category,* $F : \mathscr{C} \to \mathscr{C}'$ *be a functor such that* $F(s)$ *is an isomorphism for all* $s \in S$. *Then* F *factors uniquely through* Q.

Нульова система

Definition 1.6.6. *Let* $\mathscr{C}$ *be a triangulated category, and let* $\mathcal{N}$ *be a subfamily of* $\mathrm{Ob}(\mathscr{C})$. *One says that* $\mathcal{N}$ *is a null system if it satisfies* (N 1)–(N3) below.

(N 1) $0 \in \mathcal{N}$,
(N 2) $X \in \mathcal{N}$ *if and only if* $X[1] \in \mathcal{N}$,
(N 3) *If* $X \to Y \to Z \to X[1]$ *is a distinguished triangle, and* $X \in \mathcal{N}$, $Y \in \mathcal{N}$, *then* $Z \in \mathcal{N}$.

Now we set:

$$(1.6.4)\qquad \begin{cases} S(\mathcal{N}) = \{f : X \to Y; f \text{ is embedded into a distinguished} \\ \text{triangle } X \xrightarrow{f} Y \to Z \to X[1], \text{ with } Z \in \mathcal{N}\} \end{cases} .$$

Proposition 1.6.7. *Assume $\mathcal{N}$ is a null system. Then $S(\mathcal{N})$ is a multiplicative system.*

Proof. The property (S 1) is deduced from (N 1) and (TR 1). Let us prove (S 2). Let $X \xrightarrow{f} Y \to Z' \to X[1]$ and $Y \xrightarrow{g} Z \to X' \to Y[1]$ be two distinguished triangles, with $X' \in \mathcal{N}$, $Z' \in \mathcal{N}$. By (TR 2) there exists a distinguished triangle $X \xrightarrow{g \circ f} Z \longrightarrow Y' \longrightarrow X[1]$, and by (TR 5) there exists a distinguished triangle $Z' \to Y' \to X' \to Z'[1]$. By (N 2), (N 3) and (TR 3), we have: $Y' \in \mathcal{N}$. Hence $g \circ f \in S(\mathcal{N})$.

To prove (S 3), consider a distinguished triangle $Z \xrightarrow{g} Y \xrightarrow{k} X' \to Z[1]$, with $X' \in \mathcal{N}$, and let $f : X \to Y$. There exists a distinguished triangle

$$W \xrightarrow{h} X \xrightarrow{k \circ f} X' \longrightarrow W[1] .$$

Then by (TR 4) and (TR 3), we have a morphism of distinguished triangles:

$$\begin{array}{ccccccc} W & \xrightarrow{h} & X & \xrightarrow{k \circ f} & X' & \longrightarrow & W[1] \\ \downarrow & & \downarrow f & & \downarrow \mathrm{id}_{X'} & & \downarrow \\ Z & \xrightarrow{g} & Y & \xrightarrow{k} & X' & \longrightarrow & Z[1] \end{array} .$$

Since $X' \in \mathcal{N}$, h belongs to $S(\mathcal{N})$.

A similar proof holds by reversing the arrows.

Finally we prove (S 4). Let $f : X \to Y$ and $t : Y \to Y'$, with $t \in S(\mathcal{N})$ and $t \circ f = 0$. We shall show that there exists $s : X' \to X, s \in S(\mathcal{N})$, such that $f \circ s = 0$. Let $Z \xrightarrow{g} Y \xrightarrow{t} Y' \to Z[1]$ be a distinguished triangle, with $Z \in \mathcal{N}$. By (TR 1), (TR 3), (TR 4), there exists $h : X \to Z$ such that $f = g \circ h$. If we embed h into a distinguished triangle $X' \xrightarrow{s} X \xrightarrow{h} Z \to X'[1]$ then s will satisfy the desired properties. The proof of the converse implication is similar. □

Notation 1.6.8. Let $\mathscr{C}$ be a triangulated category and $\mathcal{N}$ a null system in $\mathscr{C}$. We write $\mathscr{C}/\mathcal{N}$ instead of $\mathscr{C}_{S(\mathcal{N})}$.

Proposition 1.6.9. *Let $\mathscr{C}$ be a triangulated category and $\mathcal{N}$ a null system.*

(i) *$\mathscr{C}/\mathcal{N}$ becomes a triangulated category by taking for distinguished triangles those isomorphic to the image of a distinguished triangle in $\mathscr{C}$.*
(ii) *Denote by Q the natural functor $\mathscr{C} \to \mathscr{C}/\mathcal{N}$.* We have *$Q(X) \simeq 0$ for $X \in \mathcal{N}$.*
(iii) *Any functor $F : \mathscr{C} \to \mathscr{C}'$ of triangulated categories such that $F(X) \simeq 0$ for all $X \in \mathcal{N}$, factors uniquely through Q.*

Похідна категорія

$\mathscr{C} = \Bbbk\text{-mod}$.

We shall apply the preceding construction to the triangulated category $\mathbf{K}(\mathscr{C})$. It is clear that:

$$\mathscr{N} = \{X \in \mathrm{Ob}(\mathbf{K}(\mathscr{C}));\ H^n(X) = 0 \text{ for any } n\} \tag{1.7.1}$$

is a null system. Note that, in view of Proposition 1.5.6, $S(\mathscr{N})$ consists of quasi-isomorphisms of $\mathbf{K}(\mathscr{C})$.

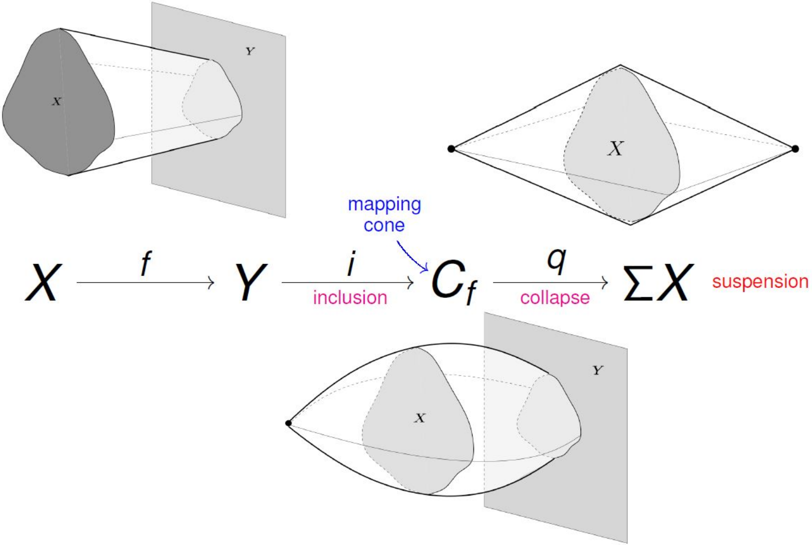
Definition 1.7.1. *We set $\mathbf{D}(\mathscr{C}) = \mathbf{K}(\mathscr{C})/\mathscr{N}$ and call $\mathbf{D}(\mathscr{C})$ the derived category of $\mathscr{C}$.*

By replacing $\mathbf{K}(\mathscr{C})$ with $\mathbf{K}^b(\mathscr{C})$ (resp. $\mathbf{K}^+(\mathscr{C})$, resp. $\mathbf{K}^-(\mathscr{C})$), we define similarly the derived categories $\mathbf{D}^b(\mathscr{C})$ (resp. $\mathbf{D}^+(\mathscr{C})$, resp. $\mathbf{D}^-(\mathscr{C})$). By Proposition 1.6.3 the functor $H^n(\cdot): \mathbf{K}(\mathscr{C}) \to \mathscr{C}$, factors through $\mathbf{D}(\mathscr{C})$. We still denote by $H^n(\cdot)$ the functor from $\mathbf{D}(\mathscr{C})$ to $\mathscr{C}$ so obtained.

Proposition 1.7.2. (i) *$\mathbf{D}^b(\mathscr{C})$ (resp. $\mathbf{D}^+(\mathscr{C})$, resp. $\mathbf{D}^-(\mathscr{C})$) is equivalent to the full subcategory of $\mathbf{D}(\mathscr{C})$ consisting of objects X such that $H^n(X) = 0$ for $|n| \gg 0$ (resp. $n \ll 0$, resp. $n \gg 0$).*

(ii) *By the composition of the functors $\mathscr{C} \to \mathbf{K}(\mathscr{C}) \to \mathbf{D}(\mathscr{C})$, $\mathscr{C}$ is equivalent to the full subcategory of $\mathbf{D}(\mathscr{C})$ consisting of objects X such that $H^n(X) = 0$ for $n \neq 0$.*

Конус відображення в топології



Конус відображення як функтор

$\mathsf{Cone} = M$ є функтором $\mathbf{dg}^{\to} \to \mathbf{dg}$, де $\to$ = категорія з двома об'єктами 0, 1 і єдиною нетотожною стрілкою $0 \to 1$.

$\mathscr{C}^{\to}$ = категорія стрілок в $\mathscr{C}$:

$\mathsf{Ob}\,\mathscr{C}^{\to} = \mathsf{Mor}\,\mathscr{C}$,

$\mathscr{C}^{\to}(f : A \to B, g : X \to Y)$

$$= \left\{ (u : A \to X, v : B \to Y) \in (\mathsf{Mor}\,\mathscr{C})^2 \;\middle|\; \begin{array}{ccc} A & \xrightarrow{f} & B \\ u\downarrow & = & \downarrow v \\ X & \xrightarrow{g} & Y \end{array} \right\}.$$

$$\mathsf{Cone}(f : A \to B) = \left(A[1] \oplus B, \begin{pmatrix} d_{A[1]} & \sigma^{-1} \cdot f \\ 0 & d_B \end{pmatrix} \right).$$

Exercise

$$\begin{pmatrix} d_{A[1]} & \sigma^{-1} \cdot f \\ 0 & d_B \end{pmatrix}^2 = 0$$

$$\mathsf{Cone}\left(\begin{array}{ccc} A & \xrightarrow{f} & B \\ u\downarrow & = & \downarrow v \\ X & \xrightarrow{g} & Y \end{array} \right) = u[1] \oplus v.$$ Чому це $\in \mathbf{dg}$?

Коротка точна послідовність комплексів індукує виділений трикутник в похідній категорії

Proposition 1.7.5. *Let $\mathscr{C}$ be an abelian category and let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in* $\mathbf{C}(\mathscr{C})$. *Let $M(f)$ be the mapping cone of f and let $\phi^n : M(f)^n = X^{n+1} \oplus Y^n \to Z^n$ be the morphism $(0, g^n)$. Then $\{\phi^n\}_n : M(f) \to Z$ is a morphism of complexes, $\phi \circ \alpha(f) = g$, and ϕ is a quasi-isomorphism.*

Proof. It is straightforward to see that ϕ is a morphism of complexes. Moreover we have an exact sequence:

$$0 \to M(\mathrm{id}_X) \xrightarrow{\gamma} M(f) \to Z \to 0$$

where γ is associated to the morphism $\mathrm{id}_X \to f$. This last morphism is described by the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array} \quad .$$

By Proposition 1.3.6 it is enough to check that $H^n(M(\mathrm{id}_X)) = 0$ for all $n \in \mathbb{Z}$. Since $M(\mathrm{id}_X)$ is zero in $\mathbf{K}(\mathscr{C})$, this is evident. $\square$

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array} \quad \overset{\mathsf{Cone}}{\longmapsto} \quad \mathsf{Cone}(1,f) = 1 \oplus f : \mathsf{Cone}\, 1_X \to \mathsf{Cone}\, f.$$

$$\exists \phi = \begin{pmatrix} 0 \\ g \end{pmatrix} : \mathsf{Cone}\, f \to Z \in \mathbf{dg} \Leftarrow \begin{pmatrix} d_{X[1]} & \sigma^{-1} \cdot f \\ 0 & d_Y \end{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} d_Z.$$

$$0 \to \mathsf{Cone}\, 1_X \xrightarrow{\mathsf{Cone}(1,f)} \mathsf{Cone}\, f \xrightarrow{\phi} Z \to 0$$

точна як $\oplus$ в $\mathbf{gr}$ послідовностей
$0 \to X[1] \xrightarrow{1} X[1] \to 0 \to 0$ і $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$.

Exercise

$\mathsf{Cone}\, 1_X$ – стягуваний ($\exists s : 1_{\mathsf{Cone}\, 1} = sd + ds$).

Тому $\mathsf{Cone}\, 1_X$ – ациклічний $\Rightarrow \phi$ – квазі-ізоморфізм $\Rightarrow \phi$ оборотний в $D(\Bbbk\text{-mod}) \Rightarrow$ в $D(\Bbbk\text{-mod})$ існує ізоморфізм трикутників

$$\begin{array}{ccccccc} X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad g \quad} & Z & \xrightarrow{\phi^{-1}\cdot\beta(f)} & X[1] \\ \Vert & & \Vert & & \phi \big\uparrow \cong & & \Vert \\ X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\alpha(f)} & \operatorname{Cone} f & \xrightarrow{\beta(f)} & X[1] \end{array}$$

$$\begin{array}{ccc} & \text{короткі точні послідовності комплексів } \Bbbk\text{-модулів} & \\ \swarrow & & \Big\downarrow H \\ \text{виділені трикутники в } D(\Bbbk\text{-mod}) & & \\ & \searrow_{H} & \\ & \text{довгі точні послідовності в } \Bbbk\text{-mod} & \end{array}$$

In the situation of Proposition 1.7.5 the distinguished triangle $X \to Y \to Z \xrightarrow{h} X[1]$ is called the distinguished triangle associated to the exact sequence $0 \to X \to Y \to Z \to 0$. Here $h = \beta(f) \circ \phi^{-1}$.

Note that the above distinguished triangle gives rise to a long exact sequence:

$$\cdots \longrightarrow H^n(X) \longrightarrow H^n(Y) \longrightarrow H^n(Z) \xrightarrow[H^n(h)]{} H^{n+1}(X) \longrightarrow \cdots$$

and $H^n(h) = -\delta$, δ being defined in Proposition 1.3.6.

Masaki Kashiwara and Pierre Schapira, Sheaves on manifolds, Grundlehren der mathematischen Wissenschaften, vol. 292, Springer-Verlag, Berlin, New York, 1990. §1.4.4–§1.7.5