1. Гомотопійна категорія категорії модулів. Навколо похідних категорій

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Категорії модулів $\mathscr{C} = \mathbb{k}$-mod. Коланцюгові комплекси

Let $\mathscr{C}$ be an additive category.

Definition 1.3.1. *A complex X in $\mathscr{C}$ consists of the data $\{X^n, \mathrm{d}_X^n\}_{n\in\mathbb{Z}}$, such that for any $n \in \mathbb{Z}$:*

(1.3.1) $\quad X^n \in \mathrm{Ob}(\mathscr{C})$, $\quad \mathrm{d}_X^n \in \mathrm{Hom}_{\mathscr{C}}(X^n, X^{n+1})$ *and* $\quad \mathrm{d}_X^{n+1} \circ \mathrm{d}_X^n = 0$.

A morphism f from a complex X to a complex Y is a sequence $\{f^n\}_{n\in\mathbb{Z}}$ of morphisms $f^n : X^n \to Y^n$, such that for any n:

(1.3.2)
$$\mathrm{d}_Y^n \circ f^n = f^{n+1} \circ \mathrm{d}_X^n \ .$$

We denote by $\mathbf{C}(\mathscr{C})$ the category of complexes of $\mathscr{C}$ thus obtained.

One often writes a complex as a sequence:

$$\cdots \longrightarrow X^{n-1} \xrightarrow[\mathrm{d}_X^{n-1}]{} X^n \xrightarrow[\mathrm{d}_X^n]{} X^{n+1} \longrightarrow \cdots \ .$$

The family $\mathrm{d}_X = \{\mathrm{d}_X^n\}_n$ is called the **differential** of the complex X. A complex X is said to be **bounded** (resp. **bounded below**, resp. **bounded above**) if $X^n = 0$ for $|n| \gg 0$ (resp. $n \ll 0$, resp. $n \gg 0$). The full subcategory of $\mathbf{C}(\mathscr{C})$ consisting of bounded complexes (resp. complexes bounded below, resp. complexes bounded above), is denoted $\mathbf{C}^b(\mathscr{C})$ (resp. $\mathbf{C}^+(\mathscr{C})$, resp. $\mathbf{C}^-(\mathscr{C})$).

We identify $\mathscr{C}$ with the full subcategory of $\mathbf{C}(\mathscr{C})$ consisting of complexes X such that $X^n = 0$ for $n \neq 0$.

Функтор зсуву $f: M \to N$ $\deg f = -q$

Definition 1.3.2. *Let k be an integer, and let $X \in \mathrm{Ob}(\mathbf{C}(\mathscr{C}))$. One defines a new complex $X[k]$ by setting:*

$$\begin{cases} X[k]^n = X^{n+k}\,, \\ \mathrm{d}^n_{X[k]} = (-1)^k \mathrm{d}^{n+k}_X\,. \end{cases} \tag{1.3.3}$$

$\deg f(x) = \deg x - \alpha$

For a morphism $f: X \to Y$ in $\mathbf{C}(\mathscr{C})$, one defines $f[k]: X[k] \to Y[k]$ by setting:

$$f[k]^n = f^{n+k}\,. \tag{1.3.4}$$

$(x)f$

The functor $[k]$ from $\mathbf{C}(\mathscr{C})$ to $\mathbf{C}(\mathscr{C})$ is called the **shift functor** of degree k.

For any graded $\Bbbk$-module M and an integer a denote by $M[a]$ the same module with the grading shifted by a: $M[a]^k = M^{a+k}$. Denote by $\sigma^a : M \to M[a]$, $M^k \ni x \mapsto x \in M[a]^{k-a}$ the "identity map" of degree $\deg \sigma^a = -a$. Write elements of $M[a]$ as $m\sigma^a$. Typically, a map is written on the right of its argument. When $f : V \to X$ is a homogeneous map of certain degree, the map $f[a] : V[a] \to X[a]$ is defined as
$f[a] = (-1)^{a \deg f} \sigma^{-a} f \sigma^a = (-1)^{af} \sigma^{-a} f \sigma^a$.

Нехай $\mathbb{K}$ - комутативне кільце з 1.
Тензорний добуток $\mathbb{K}$-модулів $(M,N)\mapsto M\otimes_{\mathbb{K}} N$ перетворює $\mathbb{K}$-mod на симетричну моноїдальну категорію. Вона замкнена зліва і справа, тобто $\exists$ $\mathbb{K}$-модулі $\underline{\mathbb{K}\text{-mod}}^r(M,N)$ і $\underline{\mathbb{K}\text{-mod}}^l(M,N)$ разом з паруваннями $ev^r: M\otimes \underline{\mathbb{K}\text{-mod}}^r(M,N)\to N$

$ev^l: \underline{\mathbb{K}\text{-mod}}^l(M,N)\otimes M\to N$ такі що

$\forall\, L\in\mathbb{K}\text{-mod}\ \forall\, \varphi: M\otimes L\to N\ \exists!\,\psi: L\to \underline{\mathbb{K}\text{-mod}}^r(M,N)$:

$$\begin{tikzcd}
M\otimes L \arrow[d, "1\otimes\psi"'] \arrow[dr, "\varphi"] & \\
M\otimes \underline{\mathbb{K}\text{-mod}}^r(M,N) \arrow[r, "ev^r"'] & N
\end{tikzcd}$$

$\forall\, L\in\mathbb{K}\text{-mod}\ \forall\, \chi: L\otimes M\to N\ \exists!\,\xi: L\to \underline{\mathbb{K}\text{-mod}}^l(M,N)$:

$$\begin{tikzcd}
L\otimes M \arrow[d, "\xi\otimes 1"'] \arrow[dr, "\chi"] & \\
\underline{\mathbb{K}\text{-mod}}^l(M,N)\otimes M \arrow[r, "ev^l"'] & N
\end{tikzcd}$$

Маємо

$\underline{\mathbb{K}\text{-mod}}^r(M,N)=\mathbb{K}\text{-mod}\,(M,N)$, $ev^r: m\otimes f\mapsto (m)f=f(m)$

$\underline{\mathbb{K}\text{-mod}}^l(M,N)=\mathbb{K}\text{-mod}\,(M,N)$, $ev^l: f\otimes m\mapsto f(m)$

Беручи $L=\underline{\mathbb{K}\text{-mod}}^r(M,N)$, $\chi=\left(L\otimes M\xrightarrow{c} M\otimes L\xrightarrow{ev^r} N\right)$

отримуємо $\xi:\underline{\mathbb{K}\text{-mod}}^r(M,N)\to \underline{\mathbb{K}\text{-mod}}^l(M,N)$ - кан. ізом.

який ототожн. з $id:\underline{\mathbb{K}\text{-mod}}(M,N)\to \underline{\mathbb{K}\text{-mod}}(M,N)$.

Градуйовані модулі

Але розглядаємо $gr = Z - grad - k - mod$ з тензорним добутком

$(M \otimes N)^c = \bigoplus_{a+b=c} M^a \otimes N^b$ і симетрією $c : M \otimes N \to N \otimes M$,

$m \otimes n \mapsto (-1)^{mn} n \otimes m.$ $\exists$ градуйовані

k-модулі $\underline{gr}^r(M,N)$ і $\underline{gr}^l(M,N)$ разом з

паруванням $ev^r : M \otimes \underline{gr}^r(M,N) \to N$

$ev^l : \underline{gr}^l(M,N) \otimes M \to N$ такі що

$\forall\ L \in .$ $\forall \varphi : M \otimes L \to N\ \exists!\ \psi : L \to \underline{gr}^r(M,N)$:

$$\begin{tikzcd}
M \otimes L \arrow[d, "1 \otimes \psi"'] \arrow[dr, "\varphi"] & \\
M \otimes \underline{gr}^r(M,N) \arrow[r, "ev^r"'] & N
\end{tikzcd}$$

$\forall\ L \in .$ $\forall \chi : L \otimes M \to N\ \exists!\ \xi : L \to \underline{gr}^l(M,N)$:

$$\begin{tikzcd}
L \otimes M \arrow[d, "\xi \otimes 1"'] \arrow[dr, "\chi"] & \\
\underline{gr}^l(M,N) \otimes M \arrow[r, "ev^l"'] & N
\end{tikzcd}$$

Внутрішні hom'и в градуйованих модулях

ВНУТРІШНІ hom – $\underline{gr}^l(M,N)$, $\underline{gr}^r(M,N)$

збігаються з $\underline{gr}(M,N)$ – Z-градуйованим k-модулем з компонентами

$\underline{gr}(M,N)^d = \{(f^j : M^j \to N^{j+d})_{j\in Z}\}$, взяттям значення

$$ev^r : m\otimes f \mapsto (m)f = f(m)$$

$$ev^l : f\otimes m \mapsto f(m)$$

Беручи $L = \underline{gr}^r(M,N)$, $X = (L\otimes M \xrightarrow{c} M\otimes L \xrightarrow{ev^r} N)$

ОТРИМУЄМО $\xi : \underline{gr}^r(M,N) \to \underline{gr}^l(M,N)$ – КАН.ОН.ІЗОМ.

ЯКИЙ ОТОТОЖН. З відображенням

$$\underline{gr}(M,N)^d \to \underline{gr}(M,N)^d, (f^j)_j \mapsto ((-1)^{jd} f^j)_j.$$

Диференціально градуйовані модулі = комплекси

розглянемо $dg = C(k-mod)$ з двома варіантами $\otimes$ на $M \otimes N$ у сенсі gr що відрізняються диференціалом:

(лівим) $d(m \otimes n) = (dm) \otimes n + (-1)^m m \otimes (dn)$ $\quad \otimes^l$

(правим) $(m \otimes n)d = m \otimes (nd) + (-1)^n (md) \otimes n$ $\quad \otimes^r$

Втім, ці моноїдальні категорії ізоморфні завдяки інволютивному функтору

$\overline{} : dg \to dg, (M,d) \mapsto \overline{(M,d)} = (M, ((-1)^j d^j)_j).$ $\quad \overline{M \otimes^l N} = \overline{M} \otimes^r \overline{N}.$

Це замкнена симетрична моноїдальна категорія.

$$\underline{dg}(M,N) = \underline{gr}(M,N) + d$$

З 4х (ізоморфних) варіантів 2:
- ліві диференціали + праві оператори
- праві диференціали + ліві оператори

не дають диференціала на $\underline{dg}(M,N)$ у вигляді градуйованого комутатора.

Але ліві диференціали + ліві оператори дають диференціал в $\underline{dg}^l(M,N)$

$$f \mapsto [d,f] = d \circ f - (-1)^f f \circ d$$

праві диференціали + праві оператори дають диференціал в $\underline{dg}^r(M,N)$

$$f \mapsto [f,d] = f \cdot d - (-1)^f d \cdot f$$

Коланцюгові відображення = циклам в $\underline{dg}(M,N)^0$

$$f : M \to N$$

Гомотопія

$f \sim 0 \Leftrightarrow \exists s : f = ds.$

Definition 1.3.3. *A morphism $f : X \to Y$ in $\mathbf{C}(\mathscr{C})$ is called homotopic to zero if there exist morphisms $s^n : X^n \to Y^{n-1}$ in $\mathscr{C}$ such that for any n:*

$$f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n \ . \tag{1.3.5}$$

One says f is **homotopic** to g if $f - g$ is homotopic to zero. We denote by $\mathrm{Ht}(X, Y)$ the subgroup of $\mathrm{Hom}_{\mathbf{C}(\mathscr{C})}(X, Y)$ consisting of morphisms homotopic to zero. One sees easily that the composition map $\mathrm{Hom}_{\mathbf{C}(\mathscr{C})}(X, Y) \times \mathrm{Hom}_{\mathbf{C}(\mathscr{C})}(Y, Z) \to \mathrm{Hom}_{\mathbf{C}(\mathscr{C})}(X, Z)$ sends $\mathrm{Ht}(X, Y) \times \mathrm{Hom}_{\mathbf{C}(\mathscr{C})}(Y, Z)$ and $\mathrm{Hom}_{\mathbf{C}(\mathscr{C})}(X, Y) \times \mathrm{Ht}(Y, Z)$ into $\mathrm{Ht}(X, Z)$. This permits to define a new category $\mathbf{K}(\mathscr{C})$ as follows.

Definition 1.3.4. *The category $\mathbf{K}(\mathscr{C})$ is defined by*

$$\begin{cases} \mathrm{Ob}(\mathbf{K}(\mathscr{C})) = \mathrm{Ob}(\mathbf{C}(\mathscr{C})) \ , \\ \mathrm{Hom}_{\mathbf{K}(\mathscr{C})}(X, Y) = \mathrm{Hom}_{\mathbf{C}(\mathscr{C})}(X, Y)/\mathrm{Ht}(X, Y) \ . \end{cases} \tag{1.3.6}$$

One defines similarly the categories $\mathbf{K}^b(\mathscr{C})$, $\mathbf{K}^+(\mathscr{C})$ and $\mathbf{K}^-(\mathscr{C})$. They are full subcategories of $\mathbf{K}(\mathscr{C})$.

Когомології

Definition 1.3.5. *For $X \in \mathrm{Ob}(\mathbf{C}(\mathscr{C}))$, one sets:*

$$Z^k(X) = \mathrm{Ker}\, d_X^k \ , \qquad B^k(X) = \mathrm{Im}\, d_X^{k-1} \ ,$$

$$H^k(X) = \mathrm{Coker}(B^k(X) \to Z^k(X)) \ .$$

One calls $H^k(X)$ the k-th cohomology of the complex X.

In other words:

(1.3.7) $$H^k(X) = \mathrm{Ker}\, d_X^k / \mathrm{Im}\, d_X^{k-1} \ .$$

Note that $H^k(\cdot)$ is an additive functor from $\mathbf{C}(\mathscr{C})$ to $\mathscr{C}$, and:

(1.3.8) $$H^k(X) = H^0(X[k]) \ .$$

If $f : X \to Y$ is homotopic to zero, then $H^k(f) : H^k(X) \to H^k(Y)$ is the zero morphism. Hence $H^k(\cdot)$ is a well-defined functor from $\mathbf{K}(\mathscr{C})$ to $\mathscr{C}$.

$$\mathbf{K}(\Bbbk\text{-mod})(X, Y) = H^0(\underline{\mathbf{dg}}(X, Y)).$$

Деякі точні послідовності

There are exact sequences:

$$X^{k-1} \longrightarrow Z^k(X) \longrightarrow H^k(X) \longrightarrow 0 \ ,$$

$$0 \longrightarrow H^k(X) \longrightarrow \operatorname{Coker}(d_X^{k-1}) \longrightarrow X^{k+1} \ ,$$

$$0 \longrightarrow Z^{k-1}(X) \longrightarrow X^{k-1} \longrightarrow B^k(X) \longrightarrow 0 \ ,$$

$$0 \longrightarrow B^k(X) \longrightarrow X^k \longrightarrow \operatorname{Coker}(d_X^{k-1}) \longrightarrow 0 \ ,$$

$$0 \to H^k(X) \to \operatorname{Coker} d_X^{k-1} \xrightarrow[d_X^k]{} Z^{k+1}(X) \longrightarrow H^{k+1}(X) \longrightarrow 0 \ . \tag{1.3.9}$$

Exercise I.9. Let $\mathscr{C}$ be an abelian category. Consider the commutative diagram with exact rows in $\mathscr{C}$:

$y \longmapsto z$ Ker γ

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' &
\end{tikzcd}$$

$x' \longrightarrow \beta(y)$

$[x']$ Coker α

Prove that there is a natural exact sequence:

$$\operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\beta \longrightarrow \operatorname{Ker}\gamma \xrightarrow{\varphi} \operatorname{Coker}\alpha \longrightarrow \operatorname{Coker}\beta \longrightarrow \operatorname{Coker}\gamma\ ,$$

so that the following diagram commutes:

$$\begin{tikzcd}
 & Y \arrow[r, "g"] & Z \\
Y \arrow[d] & \operatorname{Ker}\gamma\circ g \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \operatorname{Ker}\gamma \arrow[u] \arrow[d, "\varphi"] \\
Y' & X' \arrow[l, "f'"] \arrow[r] & \operatorname{Coker}\alpha\ .
\end{tikzcd}$$

Довга точна послідовність

Proposition 1.3.6. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathbf{C}(\mathscr{C})$. Then there exists a canonical long exact sequence in $\mathscr{C}$:*

$$\cdots \longrightarrow H^n(X) \longrightarrow H^n(Y) \longrightarrow H^n(Z) \xrightarrow[\delta]{} H^{n+1}(X) \longrightarrow \cdots,$$

more precisely, if

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of exact sequences in $\mathbf{C}(\mathscr{C})$, then all the diagrams:

$$\begin{tikzcd}
H^n(Z) \arrow[r, "\delta"] \arrow[d] & H^{n+1}(X) \arrow[d] \\
H^n(Z') \arrow[r] & H^{n+1}(X')
\end{tikzcd}$$

commute.

Доведення довгої точної послідовності

Proof. Consider the commutative diagram with exact rows:

$$
\begin{tikzcd}
 & & & H^n(Z) \arrow[d] & \\
 & \mathrm{Coker}(d_X^{n-1}) \arrow[r] \arrow[d, "d_X^n"] & \mathrm{Coker}(d_Y^{n-1}) \arrow[r] \arrow[d, "d_Y^n"] & \mathrm{Coker}(d_Z^{n-1}) \arrow[r] \arrow[d, "d_Z^n"] & 0 \\
0 \arrow[r] & Z^{n+1}(X) \arrow[r] \arrow[d] & Z^{n+1}(Y) \arrow[r] & Z^{n+1}(Z) \, . & \\
 & H^{n+1}(X) & & &
\end{tikzcd}
$$

The result then follows from (1.3.9) and Exercise I.9. The functoriality of the construction is left to the reader. □

Гомотопічні ізоморфізми. Ланцюгові комплекси. Гомології

Remark 1.3.8. Let X and Y be two objects of $\mathbf{C}(\mathscr{C})$. One sometimes says that X and Y are homotopically equivalent if they are isomorphic in $\mathbf{K}(\mathscr{C})$, that is, if there exists $f \in \mathrm{Hom}_{\mathbf{C}(\mathscr{C})}(X, Y)$ which is an isomorphism in $\mathbf{K}(\mathscr{C})$. Such an f is called a homotopy equivalence.

Notations 1.3.9. (i) Consider a sequence $\{X_n, \mathrm{d}_n^X\}_{n \in \mathbb{Z}}$ where $X_n \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{d}_n^X \in \mathrm{Hom}_{\mathscr{C}}(X_n, X_{n-1})$ and $\mathrm{d}_n^X \circ \mathrm{d}_{n+1}^X = 0$. Then we shall still say that this sequence is a complex in $\mathscr{C}$. In fact setting $X^n = X_{-n}$, $\mathrm{d}_X^n = \mathrm{d}_{-n}^X$, the sequence $\{X^n, \mathrm{d}_X^n\}$ is a complex in our previous sense.

(ii) We sometimes denote by $X^{\cdot}$ (resp. $X_{\cdot}$) a complex $\{X^n, \mathrm{d}_X^n\}$ (resp. $\{X_n, \mathrm{d}_n^X\}$). The object $\mathrm{Ker}\, \mathrm{d}_{n-1}^X / \mathrm{Im}\, \mathrm{d}_n^X$ is called the n-th holomology group of $X_{\cdot}$ and denoted by $H_n(X_{\cdot})$.

Конус відображення

Cone f

Let $\mathscr{C}$ be an additive category, and let $f : X \to Y$ be a morphism in $\mathbf{C}(\mathscr{C})$.

Definition 1.4.1. *The mapping cone of f, denoted by $M(f)$, is the object of $\mathbf{C}(\mathscr{C})$ defined as follows:*

$M(f) = X[1] \oplus Y$

$$\begin{cases} M(f)^n = X^{n+1} \oplus Y^n \,, \\ \mathrm{d}^n_{M(f)} = \begin{pmatrix} \mathrm{d}^n_{X[1]} & 0 \\ f^{n+1} & \mathrm{d}^n_Y \end{pmatrix} . \end{cases} \tag{1.4.1}$$

Recall that $\mathrm{d}^n_{X[1]} = -\mathrm{d}^{n+1}_X$.

We define the morphisms $\alpha(f) : Y \to M(f)$ and $\beta(f) : M(f) \to X[1]$ by:

$$\alpha(f)^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix} , \tag{1.4.2}$$

$$\beta(f)^n = (\mathrm{id}_{X^{n+1}}, 0) \ . \tag{1.4.3}$$

In other notation $d_{M(f)} = \begin{pmatrix} d[1] & 0 \\ f \circ \sigma^{-1} & d \end{pmatrix}$.

Перше обертання трикутника

Lemma 1.4.2. *For any $f : X \to Y$ in $\mathbf{C}(\mathscr{C})$, there exists $\phi : X[1] \to M(\alpha(f))$ such that:*

(1.4.4) ϕ *is an isomorphism in* $\mathbf{K}(\mathscr{C})$,

(1.4.5) *The diagram below commutes in* $\mathbf{K}(\mathscr{C})$:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "\alpha(f)"] \arrow[d, "\mathrm{id}_Y"] & M(f) \arrow[r, "\beta(f)"] \arrow[d, "\mathrm{id}_{M(f)}"] & X[1] \arrow[r, "-f[1]"] \arrow[d, "\phi"] & Y[1] \arrow[d, "\mathrm{id}_{Y[1]}"] \\
& Y \arrow[r, "\alpha(f)"'] & M(f) \arrow[r, "\alpha(\alpha(f))"'] & M(\alpha(f)) \arrow[r, "\beta(\alpha(f))"'] & Y[1] .
\end{tikzcd}$$

Note that such a result would not hold in $\mathbf{C}(\mathscr{C})$. Note further that ϕ is not unique

Proof. We have:

$$M(\alpha(f))^n = Y^{n+1} \oplus M(f)^n = Y^{n+1} \oplus X^{n+1} \oplus Y^n \ .$$

We define $\phi^n : X[1]^n \to M(\alpha(f))^n$ and $\psi^n : M(\alpha(f))^n \to X[1]^n$ by:

$$\phi^n = \begin{pmatrix} -f^{n+1} \\ \mathrm{id}_{X^{n+1}} \\ 0 \end{pmatrix}, \qquad \psi^n = (0, \mathrm{id}_{X^{n+1}}, 0) \ .$$

Then the lemma follows from the following observations.

(a) $\phi = (\phi^n)_n$ and $\psi = (\psi^n)_n$ are morphisms of complexes,
(b) $\psi \circ \phi = \mathrm{id}_{X[1]}$,
(c) $\phi \circ \psi$ is homotopic to $\mathrm{id}_{M(\alpha(f))}$,
(d) $\psi \circ \alpha(\alpha(f)) = \beta(f)$,
(e) $\beta(\alpha(f)) \circ \phi = -f[1]$.

All these properties, except (c), can be checked directly. To get (c) we define $s^n : M(\alpha(f))^n \to M(\alpha(f))^{n-1}$ by:

$$s^n = \begin{pmatrix} 0 & 0 & \mathrm{id}_{Y^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Then one verifies that:

$$\mathrm{id}_{M(\alpha(f))^n} - \phi^n \circ \psi^n = s^{n+1} \circ \mathrm{d}^n_{M(\alpha(f))} + \mathrm{d}^{n-1}_{M(\alpha(f))} \circ s^n \ . \qquad \square$$

+1

$$M(\alpha(f)) = Y[1] \oplus X[1] \oplus Y, \quad d_{M(\alpha(f))} = \begin{pmatrix} d[1] & 0 & 0 \\ 0 & d[1] & 0 \\ \sigma^{-1} & f \circ \sigma^{-1} & d \end{pmatrix}.$$

$$\phi = \begin{pmatrix} -f[1] \\ 1 \\ 0 \end{pmatrix}, \ \psi = (0\,1\,0), \ s = \begin{pmatrix} 0 & 0 & \sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c)

$$1_{M(\alpha(f))} - \phi \circ \psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -f[1] & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & f[1] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d[1] & 0 & 0 \\ 0 & d[1] & 0 \\ \sigma^{-1} & f \circ \sigma^{-1} & d \end{pmatrix}$$

$$+ \begin{pmatrix} d[1] & 0 & 0 \\ 0 & d[1] & 0 \\ \sigma^{-1} & f \circ \sigma^{-1} & d \end{pmatrix} \begin{pmatrix} 0 & 0 & \sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = s \circ d + d \circ s$$

завдяки $\sigma \circ d + d[1] \circ \sigma = 0$.

Виділені трикутники

One defines a **triangle** in $\mathbf{K}(\mathscr{C})$ as being a sequence of morphisms $X \to Y \to Z \to X[1]$ and a morphism of triangles as being a commutative diagram in $\mathbf{K}(\mathscr{C})$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "="] & Y \arrow[r] \arrow[d, "\psi"'] \arrow[dr, phantom, "="] & Z \arrow[r] \arrow[d, "\chi"] \arrow[dr, phantom, "="] & X[1] \arrow[d, "\phi[1]"] \\
X' \arrow[r, "f"'] & Y' \arrow[r] & Z' \arrow[r] & X'[1] .
\end{tikzcd}$$

Definition 1.4.3. *A triangle $X \to Y \to Z \to X[1]$ in $\mathbf{K}(\mathscr{C})$ is called a distinguished triangle, if it is isomorphic to a triangle $X' \xrightarrow{f} Y' \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X'[1]$, for some f in $\mathbf{C}(\mathscr{C})$.*

Властивості $\mathsf{K}(\mathscr{C})$ як триангульованої категорії

Proposition 1.4.4. *The collection of distinguished triangles in* $\mathsf{K}(\mathscr{C})$ *satisfies the following properties,* (TR 0)–(TR 5).

(TR 0) *A triangle isomorphic to a distinguished triangle is distinguished.*

(TR 1) *For any* $X \in \mathrm{Ob}(\mathsf{K}(\mathscr{C}))$, $X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$ *is a distinguished triangle.*

(TR 2) *Any* $f : X \to Y$ *in* $\mathsf{K}(\mathscr{C})$ *can be embedded in a distinguished triangle* $X \xrightarrow{f} Y \to Z \to X[1]$.

(TR 3) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *is a distinguished triangle if and only if* $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ *is a distinguished triangle.*

(TR 4) *Given two distinguished triangles* $X \xrightarrow{f} Y \to Z \to X[1]$ *and* $X' \xrightarrow{f'} Y' \to Z' \to X'[1]$, *a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

can be embedded in a morphism of triangles (not necessarily unique).

(TR 5) (octahedral axiom). *Suppose given distinguished triangles:*

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow X[1] \ ,$$

$$Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow Y[1] \ ,$$

$$X \xrightarrow{g \circ f} Z \longrightarrow Y' \longrightarrow X[1] \ ,$$

then there exists a distinguished triangle

$$Z' \to Y' \to X' \to Z'[1]$$

such that the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d] & X[1] \arrow[d, "\mathrm{id}_{X[1]}"] \\
X \arrow[r, "g \circ f"] \arrow[d, "f"'] & Z \arrow[r] \arrow[d, "\mathrm{id}_Z"] & Y' \arrow[r] \arrow[d] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\mathrm{id}_{X'}"] & Y[1] \arrow[d] \\
Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & Z'[1] \ .
\end{tikzcd}$$

Доведення властивостей $\mathbf{K}(\mathscr{C})$ як трианг. категорії

Proof. The properties (TR 0) and (TR 2) are obvious, and (TR 3) follows from Lemma 1.4.2.

Since the mapping cone of $f : 0 \to X$ is X, the triangle $0 \longrightarrow X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0[1]$ is distinguished. Applying (TR 3) we get (TR 1). Let us prove (TR 4). We may assume that $X \xrightarrow{f} Y \to Z \to X[1]$ and $X' \xrightarrow{f'} Y' \to Z' \to X'[1]$ are $X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{\alpha(f')} M(f') \xrightarrow{\beta(f')} X'[1]$, respectively. We shall construct a morphism $w : M(f) \to M(f')$ such that:

$$(1.4.4) \qquad \begin{cases} w \circ \alpha(f) = \alpha(f') \circ v \ , \\ u[1] \circ \beta(f) = \beta(f') \circ w \ . \end{cases}$$

By the definition of $\mathbf{K}(\mathscr{C})$, there exists $s^n : X^n \to Y'^{n-1}$ such that $v^n \circ f^n - f'^n \circ u^n = s^{n+1} \circ \mathrm{d}_X^n + \mathrm{d}_{Y'}^{n-1} \circ s^n$. We define $w^n : M(f)^n = X^{n+1} \oplus Y^n \to M(f')^n = X'^{n+1} \oplus Y'^n$ by:

$$w^n = \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix} .$$

Then a direct calculation shows that w is a morphism of complexes and satisfies (1.4.4).

Let us prove (TR 5). We may assume $Z' = M(f)$, $X' = M(g)$ and $Y' = M(g \circ f)$. Let us define $u : Z' \to Y'$ and $v : Y' \to X'$ by:

$$u^n : X^{n+1} \oplus Y^n \to X^{n+1} \oplus Z^n , \qquad u = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \\ 0 & g^n \end{pmatrix} ,$$

$$v^n : X^{n+1} \oplus Z^n \to Y^{n+1} \oplus Z^n , \qquad v^n = \begin{pmatrix} f^{n+1} & 0 \\ 0 & \mathrm{id}_{Z^n} \end{pmatrix} .$$

We define $w : X' \to Z'[1]$ as the composite $X' \to Y[1] \to Z'[1]$. Then the diagram in (TR 5) is commutative, and it is enough to show that $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} Z'[1]$ is a distinguished triangle. For that purpose we shall construct an isomorphism $\phi : M(u) \to X'$ and its inverse $\psi : X' \to M(u)$ such that $\phi \circ \alpha(u) = v$ and $\beta(u) \circ \psi = w$. We have:

$$M(u)^n = M(f)^{n+1} \oplus M(g \circ f)^n = X^{n+2} \oplus Y^{n+1} \oplus X^{n+1} \oplus Z^n$$

and $X'^n = M(g)^n = Y^{n+1} \oplus Z^n$. We define ϕ and ψ by:

$$\phi^n = \begin{pmatrix} 0 & \mathrm{id}_{Y^{n+1}} & f^{n+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z^n} \end{pmatrix} , \qquad \psi^n = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{X^{n+1}} \end{pmatrix} .$$

Then one checks easily that ϕ and ψ are morphisms of complexes and $\phi \circ \alpha(u) = v$, $\beta(u) \circ \psi = w$.

Октаедр

We have $\phi \circ \psi = \mathrm{id}_{X'}$. If we define:

$$s^n : M(u)^n \to M(u)^{n-1} \quad , \qquad s^n = \begin{pmatrix} 0 & 0 & \mathrm{id}_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then:

$$(\mathrm{id}_{M(u)} - \psi \circ \phi)^n = s^{n+1} \circ \mathrm{d}^n_{M(u)} + \mathrm{d}^{n-1}_{M(u)} \circ s^n \ .$$

Hence $\psi \circ \phi$ equals $\mathrm{id}_{M(u)}$ in $\mathbf{K}(\mathscr{C})$. $\square$

Remark 1.4.5. Property (TR 5) may be visualized by the following octahedral diagram: