Dold–Kan correspondence, revisited

Dold–Kan correspondence for simplicial objects

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Assume given idempotent complete additive category $\mathcal{A}$. For a simplicial object A viewed as an Ab-functor $\mathbb{Z}\Delta^{\mathrm{op}} \to \mathcal{A}$ we denote $Ad_i = A(\partial^i)^{\mathrm{op}} : A_n \to A_{n-1}$, $0 \leqslant i \leqslant n$, $As_j = A(\sigma^j)^{\mathrm{op}} : A_n \to A_{n+1}$, $0 \leqslant j \leqslant n$, $A\pi_k = A(\pi^k)^{\mathrm{op}} : A_n \to A_n$. The equivalence $S\mathcal{A} = \text{Ab-}\mathcal{C}\mathrm{at}(\mathbb{Z}\Delta^{\mathrm{op}}, \mathcal{A}) \xleftarrow[\simeq]{\text{Ab-}\mathcal{C}\mathrm{at}(\ddot{\imath}, A)} \mathrm{add}\mathcal{C}\mathrm{at}(\widehat{\mathbb{Z}\Delta^{\mathrm{op}}}, \mathcal{A}) \xrightarrow{\simeq}$ $\mathrm{add}\mathcal{C}\mathrm{at}(\mathrm{add}\,\mathrm{Ch}_{\geqslant 0}, \mathcal{A}) \simeq \text{Ab-}\mathcal{C}\mathrm{at}(\mathrm{Ch}_{\geqslant 0}, \mathcal{A}) = \mathrm{Ch}_{\geqslant 0}(\mathcal{A})$ is isomorphic to the functor

$$\begin{aligned}
&S\mathcal{A} \to \mathrm{Ch}_{\geqslant 0}(\mathcal{A}),\\
&\quad A \mapsto \big(A\pi_{k-1} \cdot Ad_k \cdot A\pi_{k-2} :\\
&\qquad \mathrm{Im}(\pi_{k-1} : A_k \to A_k) \to \mathrm{Im}(\pi_{k-2} : A_{k-1} \to A_{k-1}) \mid k \geqslant 1\big),\\
&\quad (f : A \to B) \mapsto (A\pi_{k-1} \cdot f_k \cdot B\pi_{k-1})_{k \geqslant 0}.
\end{aligned} \tag{1}$$

Let us represent this equivalence in another form.

Consider the cosimplicial topological simplex
$\Delta_{\mathrm{top}} : [n] \mapsto \Delta^n_{\mathrm{top}} = \{(x_0, \dots, x_n) \in \mathbb{R}^{[n]}_{\geqslant 0} = \mathbb{R}^{1+n}_{\geqslant 0} \mid \sum_{i=0}^n x_i = 1\}$.
It assigns to $f : [m] \to [n] \in \Delta$ the map $\Delta^m_{\mathrm{top}} \to \Delta^n_{\mathrm{top}}$ induced by the linear map

$$\Psi(f) : \mathbb{R}^{[m]} \to \mathbb{R}^{[n]}, \quad (y_0, \dots, y_m) \mapsto (x_0, \dots, x_n), \quad x_i = \sum_{j \in f^{-1} i} y_j. \tag{2}$$

View Δ^n_{top} as a cellular space, whose cells are non-degenerate faces. Associate with it the cell complex $C^n_\bullet = C(\Delta^n_{\mathrm{top}})$ over $\mathbb{Z}$. It is isomorphic to the exterior algebra $\wedge^\bullet \mathbb{Z}^{[n]} = T^\bullet \mathbb{Z}^{[n]} / (u \otimes u)_{u \in \mathbb{Z}^{[n]}}$ over $\mathbb{Z}$, equipped with the differential $d : \wedge^k \mathbb{Z}^{[n]} \to \wedge^{k-1} \mathbb{Z}^{[n]}$ which is a right derivation (that is,
$(\omega \wedge \eta) d = \omega \wedge (\eta d) + (-1)^\eta (\omega d) \wedge \eta$) determined by the map
$d : \mathbb{Z}^{[n]} \to \mathbb{Z}$, $e_i \mapsto 1$, $0 \leqslant i \leqslant n$. The functor $[n] \mapsto \wedge^\bullet \mathbb{Z}^{[n]}$,
$(f : [m] \to [n]) \mapsto \wedge^\bullet \Psi(f)$, provides a cosimplicial differential graded ring $\wedge^\bullet \mathbb{Z}^{[\bullet]}$. Here $\Psi(f) : \mathbb{Z}^{[m]} \to \mathbb{Z}^{[n]}$ is given by (2).
Exercise: Prove directly that $\wedge^\bullet \Psi(f)$ are chain maps.

Forgetting about the multiplication and the unit we get a functor $\mathbb{Z}\Delta \to \mathsf{Ch}_{\geqslant 0}(\mathsf{fAb})$, where fAb is a full subcategory of Ab consisting of free finitely generated abelian groups. Equivalently, a functor $\mathsf{Ch}_{\geqslant 0} \to \mathsf{Ab}\text{-}\mathcal{C}\mathsf{at}(\mathbb{Z}\Delta, \mathsf{fAb})$.
Since $\mathcal{A}$ is closed under direct sums, there is an action functor which we denote by $\otimes : \mathsf{fAb} \times \mathcal{A} \to \mathcal{A}$, $(\mathbb{Z}^n, X) \mapsto \mathbb{Z}^n \otimes X \overset{\text{def}}{=} X^n$, $(f = (f_{ij})_{ij} : \mathbb{Z}^n \to \mathbb{Z}^m, X) \mapsto f \otimes 1 \overset{\text{def}}{=} (f_{ij} : X \to X)_{ij} \in \mathcal{A}(X^n, X^m)$, $(\mathbb{Z}^n, g : X \to Y) \mapsto 1 \otimes g \overset{\text{def}}{=} (g^n : X^n \to Y^n)$.

Proposition

Let $A : \Delta^{\mathsf{op}} \to \mathcal{A}$ be a simplicial object of $\mathcal{A}$. Then for any $k \geqslant 0$ there exists a coend and a cokernel $\int^{[p]\in\Delta} \wedge^{1+k}\mathbb{Z}^{[p]} \otimes A_p = \operatorname{Coker}\big((As_j)_j : \bigoplus_{j=0}^{k-1} A_{k-1} \to A_k\big) = \operatorname{Im}(A\pi_{k-1} : A_k \to A_k)$. The assignment $A \mapsto \int^{[p]\in\Delta} \wedge^{1+\bullet}\mathbb{Z}^{[p]} \otimes A_p$ extends to a functor $S\mathcal{A} \to \mathsf{Ch}_{\geqslant 0}(\mathcal{A})$, isomorphic to (1).

The **coend** of a functor $F : \mathcal{C} \times \mathcal{C}^{\text{op}} \to \mathcal{D}$ is written $\int^{c \in \mathcal{C}} F(c, c)$, and comes equipped with a universal extranatural transformation with components

$$\iota_c \colon F(c, c) \to \int^{c \in \mathcal{C}} F(c, c).$$

We unwrap the definition of an extranatural transformation.

Definition

Let $F : \mathcal{C} \times \mathcal{C}^{\text{op}} \to \mathcal{D}$ be a functor. A **cowedge** $e : F \to w$ is an object w and maps $e_c : F(c, c) \to w$ for each c, such that given any morphism $f : c \to b \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(c, b) & \xrightarrow{F(f, b)} & F(b, b) \\ {\scriptstyle F(c,f)}\downarrow & & \downarrow{\scriptstyle e_b} \\ F(c, c) & \xrightarrow{\quad e_c \quad} & w \end{array}$$

Coend

Given a cowedge $e : F \to w$ and a map $f : w \to v$, we obtain a cowedge $e \cdot f : F \to v$ by composition. We define the coend as follows:

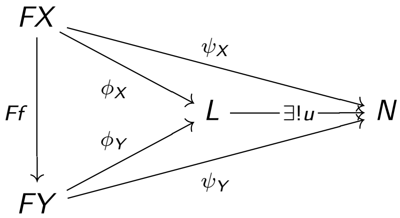
Definition

Let $F : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a functor. A **coend** of F is a universal cowedge, i.e. a cowedge $e : F \to w$ such that any other cowedge $e' : F \to w'$ factors through e via a unique map $w \to w'$.

Notation: $\int^{c \in \mathcal{C}} F(c, c)$.

Remark: coend is a particular case of a colimit.

A co-cone of a diagram $F : \mathcal{S} \to \mathcal{D}$ is an object N of $\mathcal{C}$ together with a family of morphisms $\psi_X : F(X) \to N$ for every object X of $\mathcal{S}$, such that for every morphism $f : X \to Y$ in $\mathcal{S}$, we have $\psi_Y \circ F(f) = \psi_X$.
A colimit of a diagram $F : \mathcal{S} \to \mathcal{D}$ is a co-cone (L, ϕ) of F such that for any other co-cone (N, ψ) of F there exists a unique morphism $u : L \to N$ such that $u \circ \phi_X = \psi_X$, $\forall X \in \mathcal{S}$:



Colimits are also referred to as universal co-cones. They can be characterized as initial objects in the category of co-cones from F. If a diagram F has a colimit then this colimit is unique up to a unique isomorphism.

Exercise

A coend is a colimit:

$$\int^{c \in \mathcal{C}} F(c,c) = \operatorname*{colim}_{\mathcal{C}^{\rightleftarrows}} \bar{F},$$

where $\operatorname{Ob} \mathcal{C}^{\rightleftarrows} = \operatorname{Mor} \mathcal{C}$, a morphism $(h,k) : f \to g$ of $\mathcal{C}^{\rightleftarrows}$ is the commutative square in $\mathcal{C}$

$$\begin{array}{ccc} c & \xrightarrow{\quad h \quad} & a \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle g} \\ b & \xleftarrow{\quad k \quad} & d \end{array}$$

Functor $\bar{F} : \mathcal{C}^{\rightleftarrows} \to \mathcal{D}$ takes $(f : c \to b) \mapsto F(c,b)$,
$((h,k) : f \to g) \mapsto F(h,k) : F(c,b) \to F(a,d)$.

Proof of proposition

The coend $\int^{[p]\in\Delta} \wedge^{1+k}\mathbb{Z}^{[p]} \otimes A_p$ is by definition an object $\int_k$ of $\mathcal{A}$, equipped with morphisms $i_p^k : \wedge^{1+k}\mathbb{Z}^{[p]} \otimes A_p \to \int_k$ such that for any $f : [m] \to [n] \in \Delta$ the squares

$$\begin{array}{ccc} \wedge^{1+k}\mathbb{Z}^{[m]} \otimes A_n & \xrightarrow{1\otimes A(f^{\mathrm{op}})} & \wedge^{1+k}\mathbb{Z}^{[m]} \otimes A_m \\ {\scriptstyle \wedge^{1+k}\Psi(f)\otimes 1}\downarrow & = & \downarrow{\scriptstyle i_m^k} \\ \wedge^{1+k}\mathbb{Z}^{[n]} \otimes A_n & \xrightarrow{\quad i_n^k \quad} & \int_k \end{array} \qquad (3)$$

commute and make $\int_k$ a colimit of the diagram of arrows $\wedge^{1+k}\Psi(f) \otimes 1$ and $1 \otimes A(f^{\mathrm{op}})$, where f runs over $\operatorname{Mor}\Delta$. If $n < k$, then $i_n^k = 0$. For $m = k$, $n = k-1$, particular cases of (3) take the form

$$\Big(A_{k-1} \xrightarrow{A(s_j)} A_k \xrightarrow{i_k^k} \int_k\Big) = 0, \qquad 0 \leqslant j \leqslant k-1.$$

Exercise

There exists a cokernel in $\mathcal{A}$

$$S_k \overset{\text{def}}{=} \operatorname{Coker}\Big((As_j)_{j=0}^{k-1} : \bigoplus_{j=0}^{k-1} A_{k-1} \to A_k\Big) = \operatorname{Im}(A\pi_{k-1} : A_k \to A_k),$$

$$\operatorname{coker}\Big((As_j)_{j=0}^{k-1} : \bigoplus_{j=0}^{k-1} A_{k-1} \to A_k\Big) = p_{\pi_{k-1}} : A_k \to \operatorname{Im}(A\pi_{k-1}).$$

Therefore $i_k^k = \big(A_k \xrightarrow{\bar{\imath}_k^k} S_k \xrightarrow{\eta_k} \int_k\big)$. If $n \geqslant k$, we consider $m = k$ and equation (3) for all subsets $0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n$, or for the functions $f = \chi_{i_0,\dots,i_k} : [k] \hookrightarrow [n]$, $\chi(j) = i_j$. We obtain

$$i_n^k = \sum_{0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n} \Big(\wedge^{1+k}\mathbb{Z}^{[n]} \otimes A_n \xrightarrow{\operatorname{pr}_{i_0,\dots,i_k} \otimes 1} \mathbb{Z} \otimes A_n = A_n$$

$$\xrightarrow{A(\chi_{i_0,\dots,i_k})^{\text{op}}} A_k \xrightarrow{\bar{\imath}_k^k} S_k\Big) \xrightarrow{\eta_k} \int_k .$$

Denote by $\bar{\imath}_n^k$ the sum of these compositions ending in S_k.

We shall prove the reduced version of (3), namely,

$$\begin{array}{ccc} \bigwedge^{1+k}\mathbb{Z}^{[m]} \otimes A_n & \xrightarrow{1\otimes A(f^{\mathrm{op}})} & \bigwedge^{1+k}\mathbb{Z}^{[m]} \otimes A_m \\ {\scriptstyle \wedge^{1+k}\Psi(f)\otimes 1}\downarrow & = & \downarrow {\scriptstyle \bar{\imath}^k_m} \\ \bigwedge^{1+k}\mathbb{Z}^{[n]} \otimes A_n & \xrightarrow{\quad \bar{\imath}^k_n \quad} & S_k \end{array} \tag{4}$$

If $m < k$, there is nothing to prove. Assume that $m \geqslant k$, and restrict equation to the summand $\mathbb{Z}e_{i_0} \wedge \cdots \wedge e_{i_k} \otimes A_n$ of $\bigwedge^{1+k}\mathbb{Z}^{[m]} \otimes A_n$, $0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant m$. The top-right path is

$$A_n \xrightarrow{A(\chi_{i_0,\ldots,i_k}\cdot f)^{\mathrm{op}}} A_k \xrightarrow{\bar{\imath}^k_k} S_k. \tag{5}$$

We have $\wedge^{1+k}\Psi(f)(e_{i_0} \wedge \cdots \wedge e_{i_k}) = e_{fi_0} \wedge \cdots \wedge e_{fi_k}$. If all $fi_0, \ldots fi_k \in [n]$ are distinct, then the left-bottom path is

$$A_n \xrightarrow{A(\chi_{fi_0,\ldots,fi_k})^{\mathrm{op}}} A_k \xrightarrow{\bar{\imath}^k_k} S_k$$

which coincides with (5).

If $\left([k] \xhookrightarrow{\chi_{i_0,\dots,i_k}} [m] \xrightarrow{f} [n]\right)$ is not an injection, then $\wedge^{1+k}\Psi(f)(e_{i_0} \wedge \cdots \wedge e_{i_k}) = 0$, and there are $0 \leqslant j \leqslant k-1$, $h \in \Delta$ such that $\chi_{i_0,\dots,i_k} \cdot f = \left([k] \xrightarrow{\sigma^j} [k-1] \xrightarrow{h} [n]\right)$. Therefore, (5) equals $\left(A_n \xrightarrow{A(h^{\mathrm{op}})} A_{k-1} \xrightarrow{As_j} A_k \xrightarrow{\bar{\imath}_k^k} S_k\right) = 0$, which proves (4). Hence, S_k is the colimit and is a suitable choice for $\int_k$. Note that we wished to identify S_k with $\operatorname{Im} \pi_{k-1}$. One could, in principle, find another idempotent which solves Exercise 10.

Since $\wedge^{1+\bullet}\mathbb{Z}^{[\bullet]}$ is a functor from $\Delta^{\mathrm{op}} \times \mathrm{Ch}_{\geqslant 0}$, the coend $\int^{[p]\in\Delta} \wedge^{1+\bullet}\mathbb{Z}^{[p]} \otimes A_p$ has the structure of a complex in $\mathcal{A}$. For this differential and $k \in \mathbb{N}$ the collection of morphisms given by top rows of the following diagrams is a chain map:

$$\begin{array}{ccccc}
\wedge^{1+k}\mathbb{Z}^{[m]} \otimes A_n & \xrightarrow{1\otimes A(f^{\mathrm{op}})} & \wedge^{1+k}\mathbb{Z}^{[m]} \otimes A_m & \xrightarrow{i_m^k} & \int_k \\
\downarrow{\scriptstyle d\otimes 1} & = & \downarrow{\scriptstyle d\otimes 1} & & \downarrow{\scriptstyle d} \\
\wedge^{k}\mathbb{Z}^{[m]} \otimes A_n & \xrightarrow{1\otimes A(f^{\mathrm{op}})} & \wedge^{k}\mathbb{Z}^{[m]} \otimes A_m & \xrightarrow{i_m^{k-1}} & \int_{k-1}
\end{array}$$

In fact, the right square computed on $\mathbb{Z}e_{i_0} \wedge \cdots \wedge e_{i_k} \otimes A_m$ takes the form for $\chi_{i_0,\ldots,i_k} : [k] \hookrightarrow [m]$

$$\begin{array}{ccccc}
A_m & \xrightarrow{A(\chi_{i_0,\ldots,i_k})^{\mathrm{op}}} & A_k & \xrightarrow{i_k^k} & \int_k \\
\| & = & \downarrow{\scriptstyle \sum_{j=0}^k (-1)^{k-j} Ad_j} & & \downarrow{\scriptstyle d} \\
A_m & \xrightarrow{\sum_{j=0}^k (-1)^{k-j} A(\partial_k^j \cdot \chi_{i_0,\ldots,i_k})^{\mathrm{op}}} & A_{k-1} & \xrightarrow{i_{k-1}^{k-1}} & \int_{k-1}
\end{array}$$

where $d = A\pi_{k-1} \cdot Ad_k \cdot A\pi_{k-2}$.

End of proof of proposition

Commutativity of the right square

$$\sum_{j=0}^{k}(-1)^{k-j}Ad_j \cdot A\pi_{k-2} = A\pi_{k-1} \cdot Ad_k \cdot A\pi_{k-2}$$

follows from the identity

$$\sum_{j=0}^{k}(-1)^{k-j}\pi^{k-2} \cdot \partial^j = \pi^{k-2} \cdot \partial^k \cdot \pi^{k-1}$$

which follows from some previous identity multiplied with π^{k-2} on the left.
The coend $\int^{[p]\in\Delta} \wedge^{1+k}\mathbb{Z}^{[p]} \otimes A_p$ behaves functorially with respect to A, giving a functor, which for our choice of the coend coincides with (1).