Introduction to homotopy theory

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Advantage of homotopy classifications:

- number of objects becomes discrete
- the set of objects often carries an algebraic structure (groups, rings, semigroups, etc)
- applications: almost always discrete ("quantum") invariants of something are homotopy invariants.

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Definition

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Then the maps F_0 and F_1 are called homotopic, and F is a homotopy between them.

Homotopic maps

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2) Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be continuous maps. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f' : X \to Z$.

Examples

- The map $F : \mathbb{R} \times I \to \mathbb{R}$ given by

$$F(x,t) = t\cos(x) + 2^{t^2 - x^4} + \ln(x^2 + t^2 + 2)$$

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Contractible spaces

A topological space X is contractible if the identity map id_X is homotopic to a constant map into some point.
So there exists a continuous map $F : X \times [0, 1] \to X$ such that $F_0 = \mathrm{id}_X$ and $F_1(X) = z$ for some point $z \in X$.

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Exercise

Prove that every finite tree is contractible.

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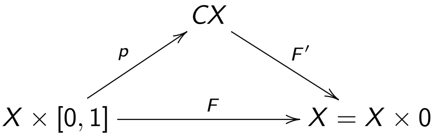

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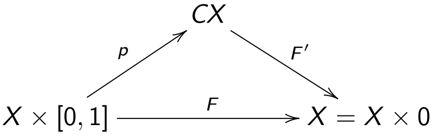

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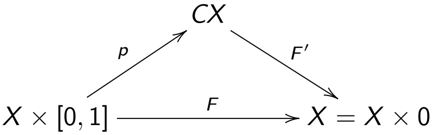

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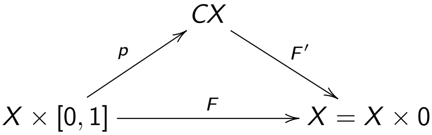

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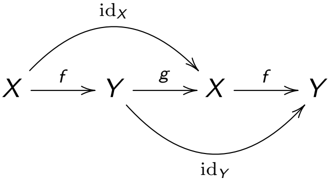
Reformulation

X is contractible whenever the base $X \times 0$ is a retract of CX.

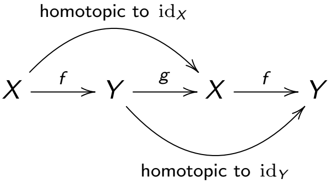
Homeomorphism

A continuous map $f : X \to Y$ is a homeomorphism if there exists a continuous map $g : Y \to X$ such that



Homotopy equivalence

A continuous map $f : X \to Y$ is a homotopy equivalence if there exists a continuous map $g : Y \to X$ such that



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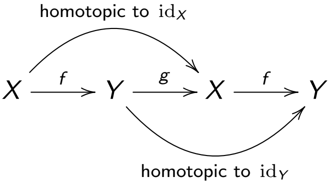

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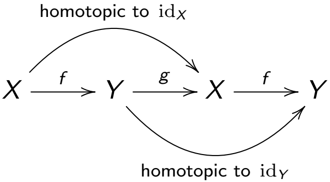

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1) Homotopy equivalence is an equivalence relation.
2) A map homotopic to a homotopy equivalence is a homotopy equivalence as well.

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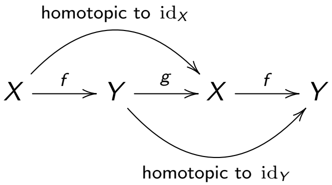

Exercises

1) Homotopy equivalence is an equivalence relation.
2) A map homotopic to a homotopy equivalence is a homotopy equivalence as well.

3) The set of self-homotopy equivalences $HE(X) \subset [X, X]$ of a topological space X constitute a subgroup in the semigroup $[X, X]$.

Homotopy equivalence, 2

Theorem

Let $F : X \to Y$ be a continuous map. Then the following conditions are equivalent:

(1) *there exists a continuous map $G : Y \to X$ such that $G \circ F : X \to X$ is homotopic to id_X (G is "right" homotopy inverse for F);*

(2) *for any topological space Z the induced map*

$$F^* : [Y, Z] \to [X, Z], \qquad F^*(g) = g \circ F : X \xrightarrow{F} Y \xrightarrow{g} Z$$

is a bijection.

Similarly, the following conditions are also equivalent:

(1′) *there exists a continuous map $G : Y \to X$ such that $F \circ G' : Y \to Y$ is homotopic to id_Y (G' is a left" homotopy inverse for F);*

(2′) *for any topological space Z the induced map*

$$F_* : [Z, X] \to [Z, Y], \qquad F_*(h) = F \circ h : Z \xrightarrow{h} X \xrightarrow{F} Y$$

is a bijection.

If both conditions (1) and (1′) hold then G and G' are homotopic.
In particular, $F : X \to Y$ **is a homotopy equivalence iff F^* and F_* are bijections for all spaces Z.**

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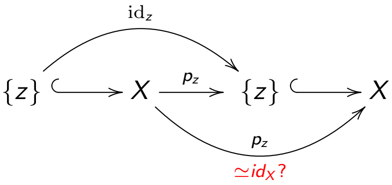
Exercise

For a topological space X TFAE:

1. (i) X is contractible (id_X is homotopic to a constant map)
2. (ii) $X \times 0$ is a retract of the cone CX
3. (iii) any two maps $f, g : Y \to X$ are homotopic
4. (iv) X is homotopy equivalent to a point

Proof (i)⇔(iv): Let $p_z : X \to \{z\}$ be a constant map.



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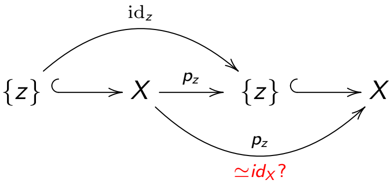

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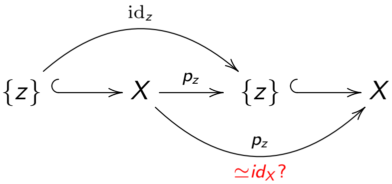

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Deformations

Let $A \subset X$ be a subset. A deformation of X into A is a homotopy $F : X \times I \to X$ such that

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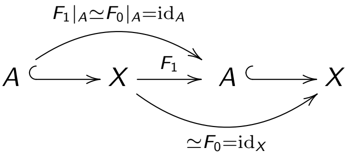

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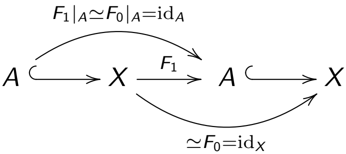

Note that $F|_{A \times I} : A \times I \to A$ is a homotopy between $F_0|_A = \mathrm{id}_A$ and $F_1|_A$.

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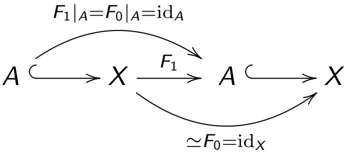

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Let X be a topological space, D^n be an n-disk, $n \geq 1$, S_+^{n-1} and S_-^{n-1} be two semispheres in the $\partial D^n = S^{n-1}$. Let $f : S_-^{n-1} \to X$ be a continuous map, and $Y = D^n \sqcup_f X$ be the topological space obtained by gluing D^n to X by the map f.

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Theorem (Chapman)

Let X and Y be two finite connected CW-complexes (polyhedrons). Then a map $f : X \to Y$ is a simple homotopy equivalence iff the map $f \times \mathrm{id} : X \times Q \to Y \times Q$ is homotopic to a homeomorpism, where $Q = \prod_{i=1}^{\infty}[0,1]$ is a Hilbert cube.

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Exercise. Let G be a topological group with unit e, and G_e be the path component of e in G. Prove that

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Exercise. Let G be a topological semigroup. Prove that

1. Partition of G into path components $\xi = \{G_\alpha\}_{\alpha \in A}$ is a congruence, that is if $a, a' \in G_\alpha$ and $b, b' \in G_\beta$, then ab and $a'b'$ belong to the same path component of G.
2. Hence the quotient set G/ξ which can be identified with $\pi_0(G, g)$ for any $g \in G$ is a semigroup, and the map $G \to \pi_0(G, g)$ is a semigroup homomorphism.

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1. Every continuous map $f : X \to Y$ induces a map $f_0 : \pi_0(X, x) \to \pi_0(Y, f(x))$ defined by $f_0[\alpha] = [f \circ \alpha]$, where $\alpha : (S^0, 0) \to (X, x)$.

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$$\alpha * \beta(t) = \begin{cases} \alpha(2t), & t \in [0, \frac{1}{2}], \\ \beta(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

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Then the operations in $\pi_1(X, x)$ is defined by $[\alpha] * [\beta] = [\alpha * \beta]$.
Exercise. 1) Prove that homotopy class of the constant map I into x is the unit of $\pi_1(X, x)$.
2) Prove that inverse to $[\alpha]$ is the homotopy class of the map $\beta : I \to X$ given by $\beta(t) = \alpha(1 - t)$.

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Proof. In fact the operation is defined by the same formula as for $\pi_k(X, x)$. □

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3. Let G be a topological group with unit e and H be its subgroup. Then $\pi_1(G, H, e)$ has a structure of a group. Moreover, the maps

$$\pi_1(G, e) \xrightarrow{j_1} \pi_1(G, H, e) \xrightarrow{\partial_1} \pi_0(H, e)$$

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1. $i : A \subset X$ be a natural inclusion. It induces a homomorphism $i_k : \pi_k(A, x) \to \pi_k(X, x)$.
2. Let $\alpha : (I^k, \partial I^k), (X, x)$ be a representative of an element of $\pi_k(X, x)$. Then, in particular, it is also a map of triples
$$\alpha : (I^k, \partial I^k, I_-^{k-1}), (X, A, x)$$
so it is also a representative of some element in $\pi_k(X, A, x)$. This gives a homomorphism
$$j_k : \pi_k(X, x) \to \pi_k(X, A, x).$$
3. Let $\alpha : (I^k, \partial I^k, I_-^{k-1}), (X, A, x)$ be a representative of an element of $\pi_k(X, A, x)$. Then, in particular, $\alpha(I_+^{k-1}) \subset A$ and $\alpha(\partial I_+^{k-1}) = x$. Thus we get the restriction map $\alpha|_{I_+^{k-1}} : (I^{k-1}, \partial I^{k-1}) \to (A, x)$ being a representative of some element $\pi_{k-1}(A, x)$.

Long exact sequence of homotopy groups

Let (X, A, x) be a triple, so $x \in A \subset X$. Let also

1. $i : A \subset X$ be a natural inclusion. It induces a homomorphism $i_k : \pi_k(A, x) \to \pi_k(X, x)$.
2. Let $\alpha : (I^k, \partial I^k), (X, x)$ be a representative of an element of $\pi_k(X, x)$. Then, in particular, it is also a map of triples
$$\alpha : (I^k, \partial I^k, I_-^{k-1}), (X, A, x)$$
so it is also a representative of some element in $\pi_k(X, A, x)$. This gives a homomorphism
$$j_k : \pi_k(X, x) \to \pi_k(X, A, x).$$
3. Let $\alpha : (I^k, \partial I^k, I_-^{k-1}), (X, A, x)$ be a representative of an element of $\pi_k(X, A, x)$. Then, in particular, $\alpha(I_+^{k-1}) \subset A$ and $\alpha(\partial I_+^{k-1}) = x$. Thus we get the restriction map $\alpha|_{I_+^{k-1}} : (I^{k-1}, \partial I^{k-1}) \to (A, x)$ being a representative of some element $\pi_{k-1}(A, x)$. In fact, $\alpha \mapsto \alpha|_{I_+^{k-1}}$ induces a so called boundary homomorphism:
$$\partial_k : \pi_k(X, A, x) \to \pi_{k-1}(A, x)$$

Long exact sequence of homotopy groups

Thus we get an infinite sequence of homomorphisms:

$$\cdots \xrightarrow{\partial_{k+1}} \pi_k(A,x) \xrightarrow{i_k} \pi_k(X,x) \xrightarrow{j_k} \pi_k(X,A,x) \xrightarrow{\partial_k}$$
$$\pi_{k-1}(A,x) \xrightarrow{i_{k-1}} \pi_{k-1}(X,x) \xrightarrow{j_{k-1}} \pi_{k-1}(X,A,x) \xrightarrow{\partial_{k-1}} \cdots$$
$$\cdots \xrightarrow{\partial_2} \pi_1(A,x) \xrightarrow{i_1} \pi_1(X,x) \xrightarrow{j_1} \pi_1(X,A,x) \xrightarrow{\partial_1}$$
$$\pi_0(A,x) \xrightarrow{i_0} \pi_0(X,x) \xrightarrow{\cong} \pi_0(X,A,x).$$

Theorem

The long sequence of homotopy groups for the triple (X,A,x) is exact.

Long exact sequence of homotopy groups

1. Compute homotopy groups $\pi_k(I, \partial I, 0)$.

Glossary

Homotopy

Glossary

Homotopy
Homotopy equivalence

Glossary

Homotopy
Homotopy equivalence
Contractible space

Glossary

Homotopy
Homotopy equivalence
Contractible space
Retract

Glossary

Homotopy
Homotopy equivalence
Contractible space
Retract
Deformation onto subspace

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Homotopy
Homotopy equivalence
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Retract
Deformation onto subspace
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